Lesson 1.1

An Election Activity

Every democratic institution must have a process by which the preferences of individuals are combined to produce a group decision. For example, the preferences of individual voters must be combined in a fair way in order to fill political offices.

An excellent way to begin an exploration of group decision making is to give the process a try. Therefore, in this lesson you will combine the preferences of the individuals in your class into a single result by a method of your own invention. Before you begin, a word of reassurance and a preview of things to come: Many important problems in election theory (and other topics in discrete mathematics) can be understood and solved without a lot of background knowledge, and mathematicians know that there is no single right way to reach a group decision.

Explore This

On a piece of paper write the names of the following soft drinks, in the order given:

Coke

Dr. Pepper

Mountain Dew

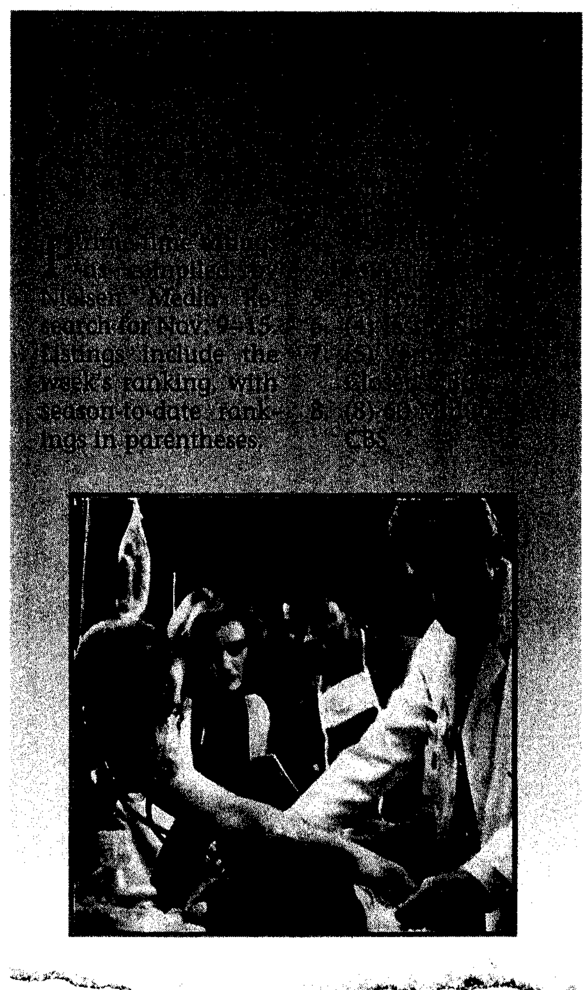
Pepsi

Sprite

CHAPTER

Election Theory

1



Throughout your life you are faced with decisions. As a student, you must decide which courses to take and how to divide your time among school work, activities, social events, and, perhaps, a job. As an adult, you will be faced with many new decisions, including whether to vote for one candidate or another.

The decisions you make are important. In the case of Nielsen television ratings, the decisions of viewers across the country determine whether a show will survive to another season. Because of the consequences of their work, organizations like Nielsen Media Research have a formidable responsibility: to combine the preferences of all the individuals in their survey into a single result and to do so in a way that is fair to all television programs.

How are the wishes of many individuals combined to yield a single result? Do the methods for doing so always treat each person fairly? If not, is it possible to improve on these methods? This chapter examines a process that is fundamental to any democratic society: group decision making. You may be surprised to learn that the study of this process is considered a part of mathematics, but the boundaries of mathematics were extended considerably in the last half of the twentieth century. Election theory is one of the most recent inclusions.

Rank the soft drinks. That is, beside the name of the soft drink you like best, write "1." Beside the name of your next favorite soft drink, write "2." Continue until you have ranked all five soft drinks.

As directed by your instructor, collect the ballots from all members of your class and share the results by, for example, writing all ballots on a chalkboard. Since everyone has written the soft drinks in the same order, you should be able to record quickly only the digits from each ballot.

Your task in this activity is to devise a method of combining the rankings of all the individuals in your class into a single class ranking . Your method should produce a first-, second-, third-, fourth-, and fifth-place soft drink for the entire class.

If you are working in a small group, your group should agree on a single method. After everyone finishes, each group or individual should present the ranking to the class and describe the method used to obtain it. Clear communication of the method used to obtain a result is important in mathematics, so everyone should strive for clarity when making the presentation.

As each group (or individual) makes its presentation, record the ranking in your notebook for use in this lesson's exercises.

Exercises

1. Did all the group rankings produced in your class have the same soft drink ranked first? If not, which soft drink was ranked first most often?

2. Repeat Exercise 1 for the soft drink ranked second.

3. Repeat Exercise 1 for the soft drink ranked third.

4. Repeat Exercise 1 for the soft drink ranked fourth.

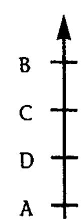
5. Repeat Exercise 1 for the soft drink ranked fifth.

6. Write a description of the method you used to achieve a group ranking. Make it clear enough that someone else could use the method. You may want to break down the method into numbered steps.

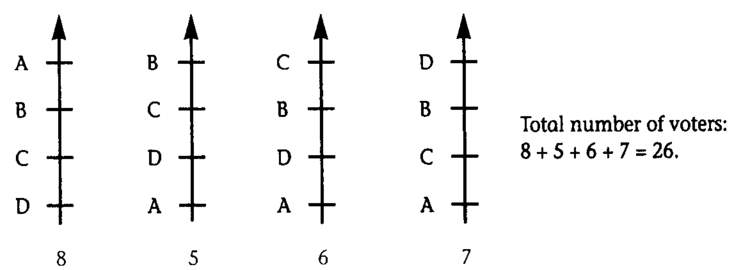
7. Did anyone in your class use a method similar to yours? Explain why you think they are similar.

8. Did your method result in any ties? How could you modify your method to break a tie?

9. Mathematicians often find it convenient to represent a situation in a compact way. A good representation conveys all the essential information about a situation. A **preference schedule** is a way to represent the preferences of one or more individuals. The preference schedule shown below displays four choices, called A, B, C, and D. This preference schedule indicates that the individual whose preference it represents ranks B first, C second, D third, and A fourth.



Since there are often several people who have the same preferences, mathematicians write the number of people or the percentage of people who expressed that preference under the schedule. The preferences in a group of 26 people are represented by the preference schedules shown in the following figure.



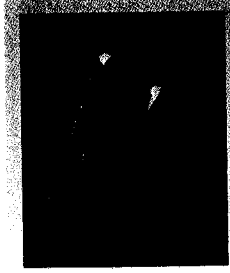
a. Apply the method you used to determine your class's soft drink ranking to this set of preferences. List the first-, second-, third-, and fourth-place rankings that your method produces. If your method cannot be applied to this set of preferences, then explain why it cannot and revise it so that it can be used here.

b. Do you think the ranking your method produces is fair? If you worked in a group, do all members of your group think the result is fair? In other words, do the first-, second-, third-, and fourth-place rankings seem reasonable, or are there reasons that one or more of the rankings seem unfair? Explain.

c. Would preference schedules be a useful way to represent the individual preferences for soft drinks among the members of your class? Explain.

10. When your class members voted, they ranked the soft drinks from first through fifth. However, voters in most U.S. elections do not get to rank the candidates. Do you think allowing voters to vote by ranking candidates would be a good practice? Explain.

Point of Interest



1996 Presidential Election Results

Candidate	Popular Vote	Electoral College Vote
Bill Clinton	47,402,357	279
Bob Dole	39,198,755	159
Ross Perot	8,085,402	0
Others	1,591,120	0

Although presidential elections in the United States are decided in the electoral college, it is rare for the winner of the popular vote to lose in the electoral college.

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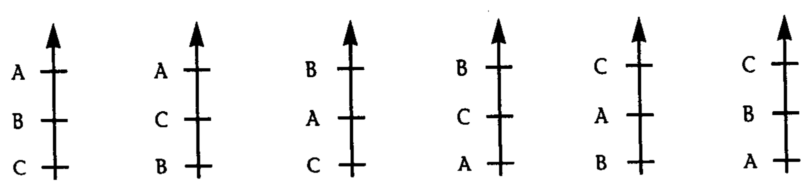
11. There are three choices in a situation that permits individuals to rank the choices. Call the choices A, B, and C. The following figure gives the six possible preferences that an individual could express.



A fourth choice, D, enters the picture. If D is attached to the bottom of each of the previous schedules, there are six schedules with D at the bottom. Similarly, there are six schedules with D third, six with D second, and six with D first, or a total of 4(6) = 24 schedules. Thus, the total number of schedules with four choices is four times the total number of schedules with three choices.

a. There are 24 possible schedules with four choices. How many are there with five choices? With six?

b. Mathematicians use symbols to represent this relationship. The symbol S_n represents the number of schedules when there are n choices. You have seen that $S_n = nS_{n-1}$. Write an English translation of the mathematical sentence $S_n = nS_{n-1}$.

12. The mathematical sentence in Exercise 11b is a **recurrence relation,** a verbal or symbolic statement that describes how one number in a list can be derived from the previous number (or numbers). If, for example, the first number in a list is 7 and the recurrence relation states that to obtain any number in the list you must add 4 to the previous number, the second number in the list is 7 + 4, or 11. This recurrence relation is stated symbolically as $T_n = 4 + T_{n-1}$. Another example of a recurrence relation is $T_n = n + T_{n-1}$. Complete the following table for the recurrence relation $T_n = n + T_{n-1}$.

n	T_n
1	3
2	2 + 3 = 5
3	
4	
5	

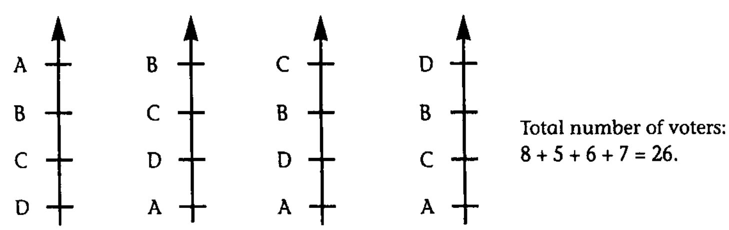
13. Complete the following table for the recurrence relation $A_n = 3 + 2A_{n-1}$.

n	A_n
1	4
2	3 + 2(4) = 11
3	
4	
5	

If the soft drink data for your class are typical, you know that the problem of establishing a group ranking is not without controversy. The reason is that there is seldom complete agreement on the correct way to achieve a group ranking. This lesson examines several common methods of determining a group ranking from a set of individual preferences. As you examine these methods, consider whether any of them are similar to the one you devised in the previous lesson.

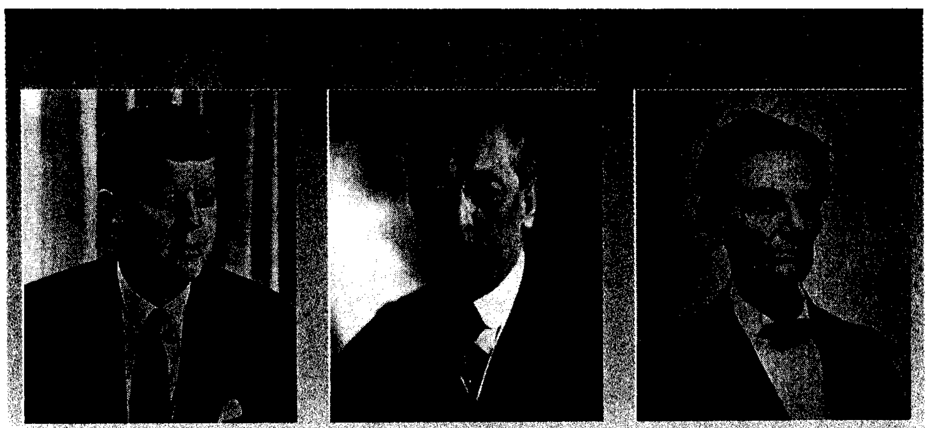
Consider the preferences of Exercise 9 of the previous lesson, which are shown again in Figure 1.1.

Many group-ranking situations, such as elections in which only one office is to be filled, require the selection of a single winner. In the set of preferences shown in Figure 1.1, choice A is ranked first on eight schedules,



Preferences of 26 voters.

more often than any other choice. If A wins on this basis, A is called the **plurality winner.** The plurality winner is based on first-place rankings only. The winner is the choice that receives the most votes. Note, however, that A is first only on about 30.8% of the schedules. Had A been first on over half the schedules, A would be a **majority winner.**



The system used to determine the president of the United States does not require that the winner receive a majority of the popular vote. Among the presidents who have served at least one term without getting a majority of the popular vote are John F. Kennedy, Woodrow Wilson, and Abraham Lincoln.

[illegible]ary officer, naval captain, mathematician, and scientist, preferred a method that assigned points to the rankings of individuals because he was dissatisfied with the plurality method.

The Borda Method

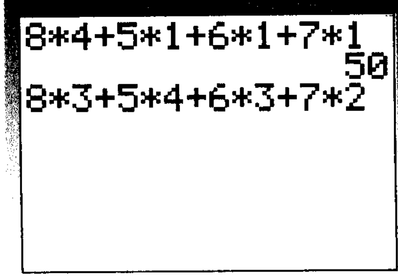
Did anyone in your class determine the soft-drink ranking by assigning points to the first, second, third, and fourth choice of each individual's preference and obtaining a point total? If so, these groups used a type of **Borda count.**

The most common way of applying the Borda method to a ranking of n choices is to assign n points to a first-place ranking, $n - 1$ to a second-place ranking, $n - 2$ to a third-place ranking, . . . , and 1 point to a last-place ranking. The group ranking is established by totaling each choice's points.

In the example of Figure 1.1, A is ranked first by 8 people and fourth by the remaining 18, so A's point total is $8(4) + 18(1) = 50$. Similar calculations give totals of 83, 69, and 58 for B, C, and D, respectively, as summarized below.

$$\begin{aligned}
A&: 8(4) + 5(1) + 6(1) + 7(1) = 50.\\
B&: 8(3) + 5(4) + 6(3) + 7(3) = 83.\\
C&: 8(2) + 5(3) + 6(4) + 7(2) = 69.\\
D&: 8(1) + 5(2) + 6(2) + 7(4) = 58.
\end{aligned}$$

In this case, the plurality winner does not fare well under the Borda system.



calculation is typed and entered, it is replayed and edited.

The Runoff Method

Many elections in the United States and other countries require a majority winner. If there is no majority winner, a runoff election between the top two candidates is held. Runoff elections are expensive because of the cost of holding another election and time-consuming because they require a second trip to the polls. However, if voters are allowed to rank the candidates, these inconveniences can be avoided.

To conduct a runoff, determine the number of firsts for each choice. In the example of Figure 1.1, A is first eight times, B is first five times, C is first six times, and D is first seven times.

Eliminate all choices except the two with the highest totals: Choices B and C are eliminated, and A and D are retained. Now consider each of the preference schedules on which the eliminated choices were ranked first. Choice B was first on the second schedule. Of the two remaining choices, A and D, D is ranked higher than A, so these 5 votes are transferred to D. Similarly, the 6 votes from the third schedule are transferred to D. The totals are now 8 for A and 7 + 5 + 6 = 18 for D, and so D is the runoff winner (see Figure 1.2).

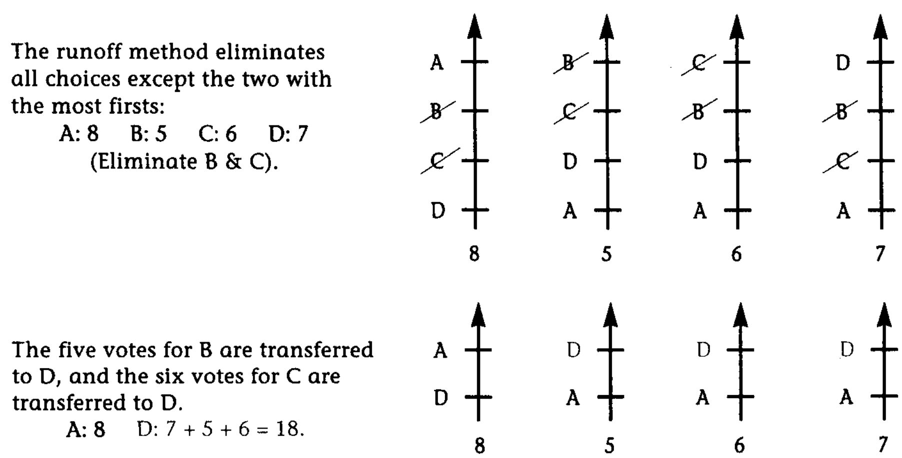


Figure 1.2 The runoff method.

The Sequential Runoff Method

Some elections, such as the voting to determine the site of the Olympic Games, are conducted by a variation of the runoff method that eliminates only one choice at a time. Although the members of the Olympic committee do not find it inconvenient to vote several times in the course of a few days, it would be impractical to do so when millions of voters are involved. Fortunately, as with the runoff method, if voters rank the candidates, they need vote only once.

In the example of Figure 1.1, B is eliminated first because it is ranked first the fewest times. The 5 first-place votes for B are transferred to C. The point totals are now 8 for A, 5 + 6 = 11 for C, and 7 for D.

There are three choices remaining. Now D's total is the smallest, so D is eliminated next. The 7 votes are transferred to the remaining choice that is ranked highest by these 7. Thus, C is given an additional 7 votes and so defeats A by 18 to 8 (see Figure 1.3).

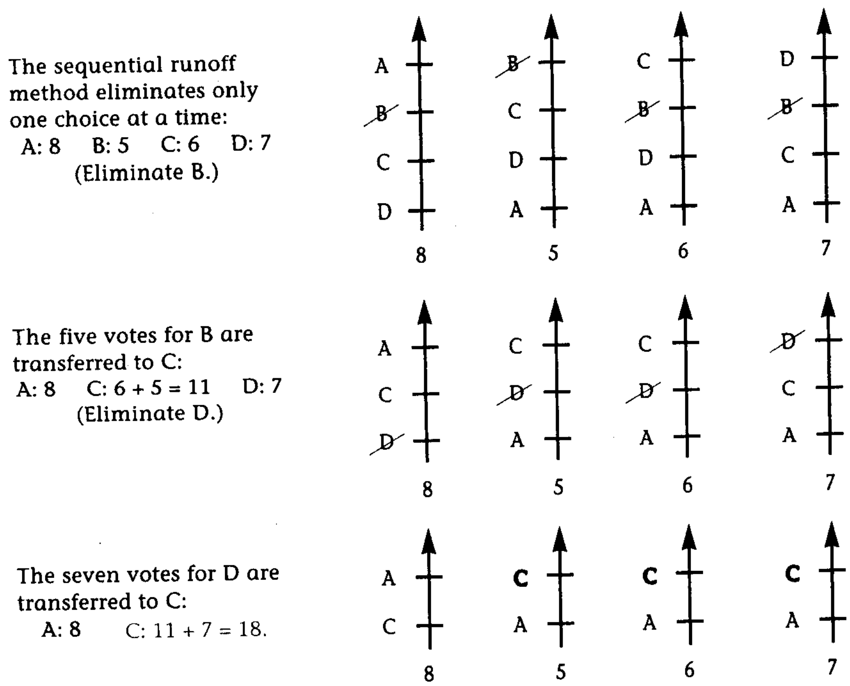


Figure 1.3 The sequential runoff method.

Exercises

1. Which soft drink is the plurality winner in your class? Is it also a majority winner? Explain.
2. Which soft drink is the Borda winner in your class?
3. Which soft drink is the runoff winner in your class?
4. Which soft drink is the sequential runoff winner in your class?

2004 Olym[illegible] to Athens

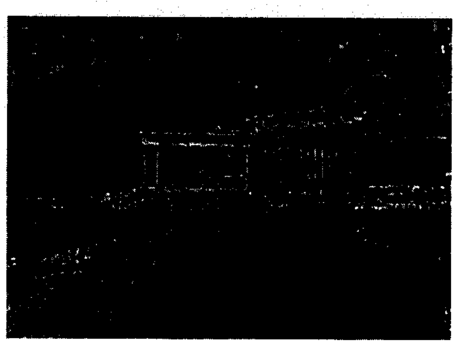
Lausanne, Switzerland, September 6, 1997

Athens took the lead in the first round of voting to decide the 2004 Olympic Games host on Friday and maintained that lead until the decisive fourth round. Athens won the right to stage the Games with 66 votes in the final round to 41 for Rome.

How the International Olympic Committee members voted:

Athens	32	38	52	66
Rome	23	28	35	41
Cape Town	16	22	20	
Stockholm	20	19		
Buenos Aires	16			

Cape Town won a tie-break against Buenos Aires 62–44 to advance to the second round.



5. For the example of Figure 1.1, determine the percentage of voters that ranked each choice first and last.

a. Enter the results in a table like the following:

Choice	First	Last
A		
B		
C		
D		

b. On the basis of these percentages only, which choice do you think would be the most objectionable to voters? The least objectionable? Explain your answers.

c. Which choice do you think best deserves to be ranked first for the group? Explain your reasoning.

d. Give at least one argument against your choice.

6. The 1998 race for governor of Minnesota had three strong candidates. The following are unofficial results from the general election.

Jesse Ventura	768,356
Norm Coleman	713,410
Hubert Humphrey III	581,497
Others	12,017

a. What percentage of the vote did the winner receive? Is the winner a majority winner?

b. What is the smallest percentage the plurality winner can receive in a race with exactly three candidates? Explain.

7. Determine the plurality, Borda, runoff, and sequential runoff winners for the following set of preferences.

A	B	C	C
D	D	B	D
C	A	D	A
B	C	A	B
16	20	12	7

8. The Borda method determines a complete group ranking, but the other methods examined in this lesson produce only a first. Each of these methods may be extended, however, to produce a complete group ranking.
   a. Describe how the plurality method could be extended to determine a second, third, and so forth. Apply this to the example in Figure 1.1 and list the second, third, and fourth that your extension produces.
   b. Describe how the runoff method could be extended to determine a second, third, and so forth. Apply this to the example in Figure 1.1 and list the second, third, and fourth that your extension produces.
   c. Describe how the sequential runoff method could be extended to determine a second, third, and so forth. Apply this to the example in Figure 1.1 and list the second, third, and fourth that your extension produces.

9. Each year the Heisman Trophy recognizes one of the country's outstanding college football players. The year 1997 marked the first time a defensive player received the award. The results of the voting follow. Each voter selects a player to rank first, another to rank second, and another to rank third.

Charles Woodson.

	1st	2nd	3rd	Points
1. Charles Woodson, Michigan	433	209	98	1,815
2. Peyton Manning, Tennessee	281	263	174	1,543
3. Ryan Leaf, Washington State	70	205	174	861
4. Randy Moss, Marshall	17	56	90	253
5. Ricky Williams, Texas	4	31	61	135
6. Curtis Enis, Penn State	3	18	20	65
7. Tim Dwight, Iowa	5	3	11	32
8. Cade McNown, UCLA	0	7	12	26
9. Tim Couch, Kentucky	0	5	12	22
10. Amos Zerouoe, West Virginia	3	1	10	21

a. How many points are awarded for a first-place vote? For a second-place vote? For a third-place vote?

b. Would the ranking produced by this system have differed if the plurality method had been used? Explain.

10. When runoff elections are used in the United States, voters do not rank the candidates and therefore must return to the polls to vote in the runoff. In some countries, such as Ireland, a method commonly called "instant runoff" is used. In an instant runoff, the voters rank the candidates and do not return to the polls. Examine the vote totals in the two runoffs shown below. Do the totals tell you anything about the merits of the instant runoff? Explain.

President of Ireland: 1997 Results

	General Election	Runoff
Mary Banotti	372,002	497,516
Mary McAleese	574,424	706,259
Derek Nally	59,529	
Adi Roche	88,423	
Dana Scallon	175,458	

U.S. House Texas District 9: 1996 Results

	General Election	Runoff
Nick Lampson	83,781	59,217
Steve Stockman	88,171	52,853
Geraldine Sam	17,886	

11. In the sequential runoff method, the number of choices on a given round is 1 less than the number of choices on the previous round. Let C_n represent the number of choices after n rounds and write this as a recurrence relation.

12. A procedure for solving a problem is called an **algorithm.** This section has presented various algorithms for determining a group ranking from individual preferences. Algorithms are often written in numbered steps in order to make them easy to apply. The following is an algorithmic description of the runoff method.

1. For each choice, determine the number of preference schedules on which the choice was ranked first.
2. Eliminate all choices except the two that were ranked first most often.
3. For each preference schedule, transfer the vote total to the remaining choice that ranks highest on that schedule.
4. Determine the vote total for the preference schedules on which each of the remaining choices is ranked first.
5. The winner is the choice ranked first on the most schedules.

a. Write an algorithmic description of the sequential runoff method.
b. Write an algorithmic description of the Borda method.

13. The number of first-, second-, third-, and fourth-place votes for each choice in an election can be described in a table, or *matrix,* as shown below.

The preferences:

A	B	C	C
D	D	B	D
C	A	D	A
B	C	A	B
20	10	12	15

The matrix:

	A	B	C	D
1st	20	10	27	0
2nd	0	12	0	45
3rd	25	0	20	12
4th	12	35	10	0

The number of points that a choice receives for first, second, third, and fourth place can be written in a matrix, as shown below.

	1st	2nd	3rd	4th
Points	4	3	2	1

A new matrix that gives the Borda point totals for each choice can be computed by writing this matrix alongside the first, as shown below.

$$[4 \quad 3 \quad 2 \quad 1] \begin{bmatrix} 20 & 10 & 27 & 0 \\ 0 & 12 & 0 & 45 \\ 25 & 0 & 20 & 12 \\ 12 & 35 & 10 & 0 \end{bmatrix}$$

The new matrix is computed by multiplying each entry of the first matrix by the corresponding entry in the first column of the second matrix and finding the sum of these products:

$$4(20) + 3(0) + 2(25) + 1(12) = 142.$$

This number is the first entry in a new matrix that gives the Borda point totals for choices A, B, C, and D:

$$\begin{matrix} & A & B & C & D \\ \text{Point totals:} & [142 & \_\_ & \_\_ & \_\_] \end{matrix}$$

a. Calculate the remaining entries of the new matrix.
b. If you have the matrix but not the preference schedules, by which methods is it possible to determine the winner? Explain.

Computer/Calculator Explorations

14. Enter the soft drink preferences of your class members into the election machine computer program that accompanies this book. Compare the results given by the computer to your answers to the first four exercises of this lesson. Resolve any discrepancies.

Projects

15. Write a short report on the history of any of the methods discussed in this section. Look into the lives of people who were influential in developing the method. Discuss factors that led them to propose the method.

16. Find at least two examples of group rankings that are currently used somewhere in the world but not discussed in this section. Describe how the group ranking is determined. Compare each new method with the methods described in this section. Are any of these new methods the same as the ones described in the lesson? What are some advantages and disadvantages of each new method?

17. Select one or more countries that are not discussed in this section and report on the methods they use to conduct elections.

[illegible] poll was [illegible] points for a [illegible] leased today. First-place vote[illegible] place votes are in pa-

1. Nebraska (32)	13–0	1,520
2. Michigan (30)	12–0	1,516
3. Florida State	11–1	1,414
4. North Carolina	11–1	1,292
5. UCLA	10–2	1,239
6. Florida	10–2	1,209
7. Kansas State	11–1	1,192
8. Tennessee	11–2	1,122
9. Washington State	10–2	1,076
10. Georgia	10–2	1,007

More Group-Ranking Methods and Paradoxes

Different methods of determining a group ranking often give different results. This fact led the Marquis de Condorcet to propose that a choice that could obtain a majority over every other choice should win.

Mathematician of Note

The Marquis de Condorcet (1743–1794) was a French mathematician, philosopher, and economist who shared an interest in election theory with his friend, Jean-Charles de Borda.

Again consider the set of preference schedules used as an example in the last lesson (see Figure 1.4).

To examine the data for a Condorcet winner, compare each choice with every other choice. For example, begin by comparing A with B, then with C, and finally with D. In the figure A is ranked higher than B on 8 schedules and lower on 18. (An easy way to see this is to cover C and D on all the schedules.) Because A cannot obtain a majority against B, A cannot be the Condorcet winner. Therefore, there is no need to compare A to C or to D.

Now consider B. You have already seen that B wins against A, so begin by comparing B with C. You can see that B is ranked higher than C on 8 + 5 + 7 = 20 schedules and lower than C on 6.

Now compare B with D. You see B is ranked higher than D on 8 + 5 + 6 = 19 schedules and lower than D on 7. Therefore, B can obtain a majority over each of the other choices and so is the Condorcet winner.

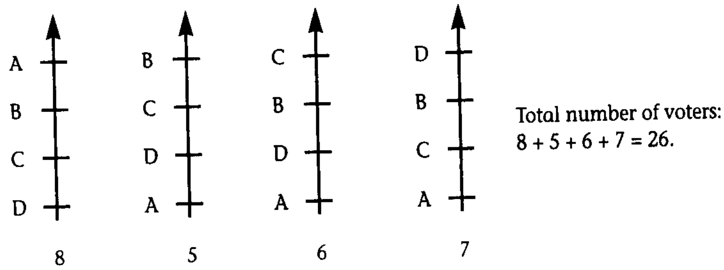


Figure 1.4 Preferences of 26 voters.

Since B is the Condorcet winner, it is unnecessary to make comparisons between C and D. Although all comparisons do not have to be made, it can be helpful to organize them in a table:

	A	B	C	D
A		L	L	L
B	W		W	W
C	W	L		W
D	W	L	L	

To see how a choice does in one-on-one contests, read across the row associated with that choice. You see A, for example, loses in one-on-one contests with B, C, and D.

Although the Condorcet method may sound ideal, it sometimes fails to produce a winner. Consider the set of schedules shown in Figure 1.5.

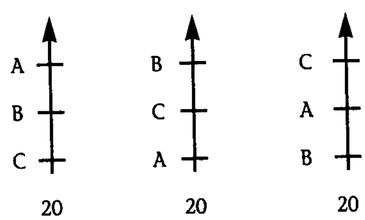


Figure 1.5 Preferences of 60 voters.

Notice that A is preferred to B on 40 of the 60 schedules but that A is preferred to C on only 20. Although C is preferred to A on 40 of the 60, C is preferred to B on only 20. Therefore there is no Condorcet winner.

You might expect that if A is preferred to B by a majority of voters and B is preferred to C by a majority of voters, then a majority of voters would prefer A to C. But the example shows that this need not be the case.

In other mathematics classes you have learned that many relationships are transitive. The relation "greater than" (>), for example, is transitive because if $a > b$ and $b > c$, then $a > c$.

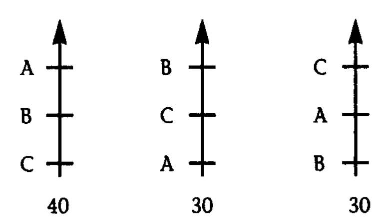
You have just seen that group-ranking methods may violate the transitive property. Because this intransitivity seems contrary to intuition, it is known as a **paradox**. This particular paradox is sometimes referred to as the **Condorcet paradox**. There are other paradoxes that can occur with group-ranking methods, as you will see in this lesson's exercises.

Exercises

1. Determine the Condorcet winner in the soft drink ballot your class conducted in Lesson 1.1.

2. Propose a method for resolving situations in which there is no Condorcet winner.

3. In a system called **pairwise voting**, two choices are selected and a vote taken. The loser is eliminated, and a new vote is taken in which the winner is paired against a new choice. This process continues until all choices but one have been eliminated. An example of the use of pairwise voting occurs in legislative bodies in which bills are considered two at a time. The choices in the set of preferences shown in the following figure represent three bills being considered by a legislative body.



   a. Suppose you are responsible for deciding which two will appear on the agenda first. If you strongly prefer bill C, which two bills would you place on the agenda first? Why?
   b. Is it possible to order the voting so that some other choice will win? Explain.

4. A panel of sportswriters is selecting the best football team in a league, and the preferences are distributed as follows.

A	B	C
B	A	B
C	C	A
52	38	10

a. Determine the winning team using a 3-2-1 Borda count.

b. The 38 who ranked B first and A second decide to lie in order to improve the chances of their favorite and so rank C second. Determine the winner using a 3-2-1 Borda count.

5. When people decide to vote differently from the way they feel about the choices, they are said to be *voting insincerely*. People are often encouraged to vote insincerely because they have some idea of the election's result beforehand. Explain why advance knowledge is possible.

6. Many political elections in the United States are decided by the plurality method. Construct a set of preferences with three choices in which the plurality method would encourage insincere voting. Identify the group of voters that would be encouraged to vote insincerely and explain the effect of their doing so on the election.

7. Many people consider the plurality method unfair because it sometimes produces a winner that a majority of voters do not like.

a. What percentage of voters rank the plurality winner last in the preferences shown below?

A	C	B	D
B	B	D	E
D	E	E	C
C	D	C	B
E	A	A	A
20	18	15	12

b. Runoffs are sometimes used to avoid the selection of a controversial winner. Is the runoff winner an improvement over the plurality winner in this set of preferences? Explain.
c. Do you consider the sequential runoff winner an improvement over the plurality and runoff winners? Explain.

8. a. Use a runoff to determine the winner in the set of preferences shown below.

A	C	B	B
B	A	C	A
C	B	A	C
38	30	25	7

b. In some situations, votes are made public. For example, people have the right to know how their elected officials vote on issues. Suppose these schedules represent such a situation. Because they expect to receive some favors from the winner and because they expect A to win, the seven voters associated with the last schedule decide to change their preferences from

B	to	A
A		B
C		C

and to "go with the winner." Conduct a new runoff and determine the winner.
c. Explain why the results are a paradox.

9. a. Use a 4-3-2-1 Borda count to determine a group ranking for the following set of preferences.

A	B
C	D
B	A
D	C
50	40

b. These preferences represent the ratings of four college athletic teams, and team C has been disqualified because of a recruiting violation. Write the schedules with team C removed and use a 3-2-1 Borda count to determine the group ranking.

c. Explain why these results are a paradox.

10. Write a brief summary of the five methods of achieving a group ranking (plurality, Borda, runoff, sequential runoff, and Condorcet) that have been discussed in this and the previous lesson. Include at least one example of why each method can lead to unfair results.

11. The Condorcet method requires that in theory each choice be compared with every other one, although in practice many of the comparisons do not have to be made in order to determine the winner. Consider what the number of comparisons would be if every comparison were made.

Mathematicians sometimes find it helpful to represent the choices and comparisons visually. If there are only two choices, a single comparison is all that is necessary. In the diagram that follows, the choices are represented by points, or *vertices,* and the comparisons by line segments, or *edges.*

a. Add a third choice, C, to the diagram. Connect it to A and to B to represent the additional comparisons. How many new comparisons are there? What is the total number of comparisons that must be made?

b. Add a fourth choice, D, to the diagram. Connect it to each of A, B, and C. How many new comparisons are there, and what is the total number of comparisons?

c. Add a fifth choice to the diagram and repeat. Then add a sixth choice and repeat. Complete the following table.

Number of Choices	Number of New Comparisons	Total Number of Comparisons
1	0	0
2	1	1
3		
4		
5		
6		

12. Let C_n represent the total number of comparisons necessary when there are n choices. Write a recurrence relation that expresses the relationship between C_n and C_{n-1}.

Computer/Calculator Explorations

13. Use the preference schedule program that accompanies this book to find a set of preferences with at least four choices that demonstrate the same paradox found in Exercise 8 when the sequential runoff method is used.

14. Use the preference schedule program to enter several schedules with five choices. Use the program's features to alter your data in order to produce a set of preferences with several different winners. Can you find a set of preferences with five choices and five winners? If so, what is the minimum number of schedules with which this can be done? Explain.

Projects

15. Research and report on paradoxes in mathematics. Try to determine whether the paradoxes have been satisfactorily resolved.

16. Research and report on paradoxes outside mathematics. In what way have these paradoxes been resolved?

17. Select an issue of current interest in your community that involves more than two choices. Have each member of your class vote by writing a preference schedule. Compile the preferences and determine the winner by five different methods.

18. Investigate the contributions of Charles Dodgson (Lewis Carroll) to election theory. Was he responsible for any of the group-ranking procedures you have studied? What did he suggest doing when the Condorcet method fails to produce a winner?

19. Investigate the system your school uses to determine academic rankings of students. Is it similar to any of the group-ranking procedures you have studied? If so, could it suffer from any of the same problems? Propose another system and discuss why it might be better or worse than the one currently in use.

20. Investigate elections in your school (class officers, officers of organizations, homecoming royalty, and so forth). Report on the type of voting and the way winners are chosen. Recommend alternative methods and explain why you think the methods you recommend would be more fair.

Arrow's Conditions and Approval Voting

Paradoxes, unfair results, and insincere voting are some of the problems that have caused people to look for better ways to reach group decisions. In this lesson you will learn of some recent and important work that has been done to improve the group-ranking process. First, consider an example involving pairwise voting.

Ten representatives of the language clubs at Central High School are meeting to select a location for the clubs' annual joint dinner. The committee must choose among a Chinese, French, Italian, or Mexican restaurant (see Figure 1.6).

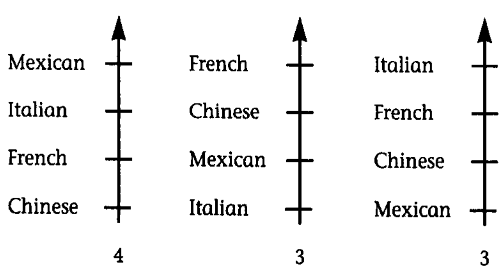


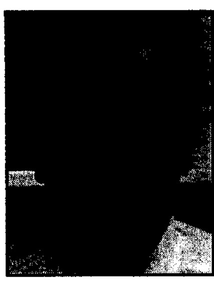
Figure 1.6 Preferences of 10 students.

Racquel suggests that because the last two dinners have been held at Mexican and Chinese restaurants, this year's dinner should be at either an Italian or a French restaurant. The group votes 7 to 3 in favor of the Italian restaurant.

Martin, who doesn't like Italian food, says that the community's newest Mexican restaurant has an outstanding reputation. He proposes that the group choose between Italian and Mexican. The other members agree and vote 7 to 3 to hold the dinner at the Mexican restaurant.

Sarah, whose parents own a Chinese restaurant, says that she can obtain a substantial discount for the event. The group votes between the Mexican and Chinese restaurants and selects the Chinese by a 6 to 4 margin.

Look carefully at the group members' preferences. Note that French food is preferred to Chinese by all, yet the voting has resulted in the selection of the Chinese restaurant!



Kenneth Arrow (1921–) received a degree in mathematics before turning to economics. His application of mathematical methods to election theory brought him worldwide recognition.

In 1951, paradoxes such as this led Kenneth Arrow, a U.S. economist, to formulate a list of five conditions that he considered necessary for a fair group-ranking method. These fairness conditions today are known as **Arrow's conditions.**

One of Arrow's conditions says that if every member of a group prefers one choice to another, then the group ranking should do the same. According to this condition, the choice of the Chinese restaurant when all members rated French food more favorably is unfair. Thus, Arrow considers pairwise voting a flawed group-ranking method.

Arrow's Conditions

1. Nondictatorship: The preferences of a single individual should not become the group ranking without considering the preferences of the others.
2. Individual Sovereignty: Each individual should be allowed to order the choices in any way and to indicate ties.
3. Unanimity: If every individual prefers one choice to another, then the group ranking should do the same. (In other words, if

every voter ranks A higher than B, then the final ranking should place A higher than B.)

4. Freedom from Irrelevant Alternatives: The winning choice should still win if one of the other choices is removed. (The choice that is removed is known as an irrelevant alternative.)

5. Uniqueness of the Group Ranking: The method of producing the group ranking should give the same result whenever it is applied to a given set of preferences. The group ranking should also be transitive.

Arrow inspected the common methods of determining a group ranking for adherence to his five conditions. He also looked for new methods that would meet all five. After doing so, he arrived at a surprising conclusion.

In this lesson's exercises, you will examine a number of group-ranking methods for their adherence to Arrow's conditions. You will also learn Arrow's surprising result.

Exercises

1. Your teacher decides to order soft drinks for your class on the basis of the soft drink vote conducted in Lesson 1.1 but, in so doing, selects the preference schedule of a single student (the teacher's pet). Which of Arrow's conditions are violated by this method of determining a group ranking?

2. Instead of selecting the preference schedule of a favorite student, your teacher places all the individual preferences in a hat and draws one. If this method were repeated, would the same group ranking result? Which of Arrow's conditions does this method violate?

3. Do any of Arrow's conditions require that the voting mechanism include a secret ballot? Is a secret ballot desirable in all group-ranking situations? Explain.

4. Examine the paradox demonstrated in Exercise 9 of Lesson 1.3 on pages 22 and 23. Which of Arrow's conditions are violated?

5. Construct a set of preference schedules with three choices, A, B, and C, showing that the plurality method violates Arrow's fourth condition.

In other words, construct a set of preferences in which the outcome between A and B depends on whether C is on the ballot.

6. There are often situations in which insincere voting results. Do any of Arrow's conditions state that insincere voting should not be part of a fair group-ranking method? Explain.

7. Suppose that there are only two choices in a list of preferences and that the plurality method is used to decide the group ranking. Which of Arrow's conditions could be violated? Explain.

8. After failing to find a group-ranking method for three or more choices that always obeyed all his fairness conditions, Arrow began to suspect that such a method might not exist. He applied logical reasoning to the problem and proved that no method, known or unknown, can always obey all five conditions. In other words, any group-ranking method will violate at least one of Arrow's conditions in some situations.

   Arrow's proof demonstrates how mathematical reasoning can be applied to areas outside mathematics. This and other achievements earned Arrow the 1972 Nobel Prize in economics.

   Although Arrow's work means that a perfect group-ranking method will never be found, it does not mean that current methods cannot be improved. Recent studies have led several experts to recommend a system called **approval voting.**

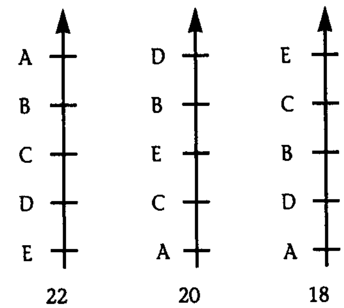
   In approval voting, you may vote for as many choices as you like, but you do not rank them. You mark all those of which you approve. For example, if there are five choices, you may vote for as few as none or as many as five.

   a. Write a soft drink ballot like the one you used in Lesson 1.1. Place an "X" beside each of the soft drinks you find acceptable. At the direction of your instructor, collect the ballots from each member of your class. Count the number of votes for each soft drink and determine the winner.
   b. Determine a complete group ranking.
   c. Was the approval winner the same as the earlier plurality winner in your class?
   d. How does the group ranking in part b compare with the earlier Borda ranking?

9. Examine Exercise 4 of Lesson 1.3 on page 21. Would any members of the panel of sportswriters be encouraged to vote insincerely if approval voting were used? Explain.

10. What is the effect on a group ranking of casting approval votes for all choices? Of casting approval votes for none of the choices?

11. The voters whose preferences are represented below all feel strongly about their first choices but are not sure about their second and third choices. They all dislike their fourth and fifth choices. Since the voters are unsure about their second and third choices, they flip coins to decide whether to give approval votes to their second and third choices.



a. Assuming the voters' coins come up heads about half the time, how many approval votes would you expect each of the five choices to get? Explain your reasoning.

b. Do the results seem unfair to you in any way? Explain.

12. Approval voting offers a voter many choices. If there are three candidates for a single office, for example, the plurality system offers the voter four choices: vote for any one of the three candidates or for none of them. Approval voting permits the voter to vote for none, any one, any two, or all three.

To investigate the number of ways in which you can vote under approval voting, consider a situation with two choices, A and B. You can represent voting for none by writing { }, voting for A by writing {A}, voting for B by writing {B}, and voting for both by writing {A, B}.

a. List all the ways of voting under the approval system when there are three choices.

b. List all the ways of voting under the approval system when there are four choices.

c. Generalize the pattern by letting V_n represent the number of ways of voting under the approval system when there are n choices and writing a recurrence relation that describes the relationship between V_n and V_{n-1}.

13. Listing all the ways of voting under the approval system can be difficult if not approached systematically. The following algorithm describes one way of finding all the ways of voting for two choices. The results are shown applied to a ballot with five choices, A, B, C, D, and E.

	List 1	List 2
1. List all choices in order in List 1.	A B C D E	
2. Draw a line through the first choice in List 1 that doesn't already have a line drawn through it and write it as many times in List 2 as there are choices in List 1 without lines through them.	~~A~~ B C D E	A A A A
3. Beside each item you wrote in List 2 in step 2, write a choice in List 1 that does not have a line through it.		AB AC AD AE
4. Repeats steps 2 and 3 until each choice has a line through it. The items in the second list show all the ways of voting for two items.		

Write an algorithm that describes how to find all the ways of voting for three choices. You may use the results of the previous algorithm to begin the new one.

14. Many patterns can be found in the various ways of voting when the approval system is used. The following table shows the number of ways of voting for exactly one item when there are several choices on the ballot. For example, in Exercise 12, you listed all the ways of voting when there are three choices on the ballot. Three of these, {A}, {B}, and {C}, are selections of one item.

Number of Choices on the Ballot	Number of Ways of Selecting Exactly One Item
1	1
2	2
3	3
4	—
5	—

Complete the table.

15. Let $V1_n$ represent the number of ways of selecting exactly one item when there are n choices on the ballot and write a recurrence relation that expresses the relationship between $V1_n$ and $V1_{n-1}$.

16. The following table shows the number of ways of voting for exactly two items when there are from one to five choices on the ballot. For example, your list in Exercise 12 shows that when there are three choices on the ballot, there are three ways of selecting exactly two items: {A, B}, {A, C}, and {B, C}.

Number of Choices on the Ballot	Number of Ways of Selecting Exactly One Item
2	1
3	3
4	—
5	—

Complete the table.

17. Let $V2_n$ represent the number of ways of selecting exactly two items when there are n choices on the ballot and write a recurrence relation that expresses the relationship between $V2_n$ and $V2_{n-1}$. Can you find more than one way to do this?

Computer/Calculator Explorations

18. Design a computer program that lists all possible ways of voting when approval voting is used. Use the letters A, B, C, . . . to represent the choices. The program should ask for the number of choices and then display all possible ways of voting for one choice, two choices, and so forth.

Projects

19. Investigate the number of ways of voting under the approval system for other recurrence relations (see Exercises 14 through 17). For example, in how many ways can you vote for three choices, four choices, and so forth?

20. Arrow's result is an example of an impossibility theorem. Investigate and report on other impossibility theorems.

21. Research and report on Arrow's theorem. The theorem is usually proved by an indirect method. What is an indirect method? How is it applied in Arrow's case?

Lesson 1.5

Weighted Voting and Voting Power


International Inc said Monday they will vote against a proposed merger with Nation Media Corp and instead have offered their own merger proposal.

The shareholders, Michael Blake and Brian Danzis, who control about 200,000 shares, said their VV Acquisition

and ValueVision have agreed to merge in a stock swap in which ValueVision holders would get 1.19 shares of a new holding company for each share exchanged and National Media holders would get one share for each share swapped.

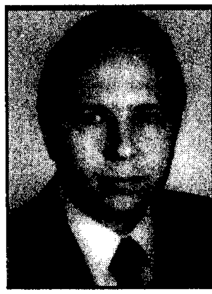
The first four lessons of this chapter examined situations in which all voters are considered equals. In some voting situations there are voters who have more votes than others; essentially, the vote of some voters carries more weight than the vote of others. This lesson examines such situations, beginning with a simple example.

A small high school has 110 students. Because of recent growth in the size of the community, the sophomore class is quite large. It has 50 members, and the junior and senior classes each have 30 members.

The school's student council is composed of a single representative from each class. Each of the three members is given a number of votes proportionate to the size of the class represented. Accordingly, the sophomore representative has five votes, and the junior and senior representatives each have three. The passage of any issue that is before the council requires a simple majority of six votes.

The student council's voting procedure is an example of **weighted voting**. Weighted voting occurs whenever some members of the voting body have more votes than others have.

Weighted voting is fairly common in the United States. For example, it is used in local government in some parts of the country and in corporate shareholder meetings. In recent years, several people have questioned whether weighted voting is fair. Among them is John Banzhaf III, a law professor at George Washington University who has initiated several legal actions against weighted voting procedures used in local government.



Mathematician of Note

John Banzhaf III, a law professor who also holds an engineering degree, is a well-known consumer rights advocate.

To understand Banzhaf's reasoning, consider the number of ways that voting on an issue could occur in the student council example.

It is possible that the issue is favored by none of the members, one of them, two of them, or all three. In which cases would the issue be passed? The following list gives all possible ways of voting for an issue and the associated number of votes.

{; 0} {So; 5} {Jr; 3} {Sr; 3} {So, Jr; 8} {So, Sr; 8} {Jr, Sr; 6} {So, Jr, Sr; 11}

For example, {Jr, Sr; 6} indicates that the junior and senior representatives could vote for an issue and that they have a total of six votes between them.

Each of these collections of voters is known as *a coalition.* Those with enough votes to pass an issue are known as **winning coalitions.** The winning coalitions in this example are those with six or more votes and are listed below along with their respective vote totals.

{So, Jr; 8} {So, Sr; 8} {Jr, Sr; 6} {So, Jr, Sr; 11}

The last winning coalition is different from the other three in one important respect: If any one of the members decides to vote differently, the coalition will still win. No one member is essential to the coalition. Banzhaf reasoned that the only time a voter has power is when the voter belongs to a coalition that needs the voter in order to pass an issue. The coalitions for which at least one member is essential are

{So, Jr; 8}   {So, Sr; 8}   {Jr, Sr; 6}

Notice that the sophomore representative is essential to two of the coalitions. This is also true of the junior and senior representatives. In other words, about the same number of times, each of the representatives can be expected to cast a key vote in passing an issue.

A paradox: Although the votes have been distributed to give greater power to the sophomores, the outcome is that all members have the same amount of power!

Since distributing the votes in a way that reflects the distribution of the population does not result in a fair distribution of power, mathematical procedures can be used to find a way to measure actual power when weighted voting is used.

A measure of the power of a member of a voting body is called a **power index**. In this lesson, a voter's power index is the number of winning coalitions to which the voter is essential. For example, in the student council situation, each representative is essential to two winning coalitions and thereby has a power index of 2, as do the junior and senior representatives.

A Power Index Algorithm

1. List all coalitions of voters that are winning coalitions.
2. Select any voter, and record a 0 for that voter's power index.
3. From the list in step 1, select a coalition of which the voter selected in step 2 is a member. Subtract the number of votes the voter has from the coalition's total. If the result is less than the number of votes required to pass an issue, add 1 to the voter's power index.
4. Repeat step 3 until you have checked all coalitions for which the voter chosen in step 2 is a member.
5. Repeat steps 2 through 4 until all voters have been checked.

Exercises

1. Consider a situation in which A, B, and C have 3, 2, and 1 votes, respectively, and in which 4 votes are required to pass an issue.
   a. List all possible coalitions and all winning coalitions.
   b. Determine the power index for each voter.
   c. If the number of votes required to pass an issue is increased from 4 to 5, determine the power index of each voter.

2. In a situation with three voters, A has 7 votes, B has 3, and C has 3.
   a. Determine the power index of each voter.
   b. A *dictator* is a member of a voting body who has all the power. A *dummy* is a member who has no power. Are there any dictators or dummies in this situation?

3. The student council example in this lesson depicted a situation with three voters that resulted in equal power for all three. In Exercises 1 and 2, power was distributed differently. Find a distribution of votes that results in a power distribution among three voters that is different from the ones you have already seen. How many new power distributions in situations with three voters can you find?

4. In this lesson's student council example, can the votes be distributed so that the members' power indices are proportionate to the class sizes? Explain.

5. In this lesson's student council example, suppose that the representatives of the junior and senior classes always differ on issues and never vote alike. Does this make any practical difference in the power of the three representatives? Explain.

6. (See Exercise 12 of Lesson 1.4 on page 29.) Let C_n represent the number of coalitions that can be formed in a group of n voters. Write a recurrence relation that describes the relationship between C_n and C_{n-1}.

7. One way to determine all winning coalitions in a weighted voting situation is to work from a list of all possible coalitions. Use A, B, C, and D to represent the individuals in a group of four voters and list all possible coalitions.

8. Weighted voting is commonly used to decide issues at meetings of corporate stockholders. Each member is given one vote for each share of stock held.
   a. A company has four stockholders: A, B, C, and D. They own 26%, 25%, 25%, and 24% of the stock, respectively, and more than 50%

of the vote is needed to pass an issue. Determine the power index of each stockholder. Use your results from Exercise 7 as an aid.

b. Another company has four stockholders. They own 47%, 41%, 7%, and 5% of the stock. Find the power index of each stockholder.

c. Compare the percentage of stock owned by the smallest stockholder in parts a and b. Do the same for the power index of the smallest stockholder in each case.

Nassau Districting Ruled Against Law

NEW YORK TIMES, January 15, 1970

Albany—The Court of Appeals ruled today that the present "weighted voting" plan of the Nassau County Supervisors was unconstitutional but that a new plan was not necessary until after the 1970 federal census.

In a unanimous opinion, the state's highest court said the county's present charter provision is a clear violation of the one-man-one-vote principle, in that it specifically denied the town of Hempstead representation that reflected its population.

The town, the court pointed out, constituted 57.12 percent of the county's population, but because of the weighted voting plan its representatives on the board could cast only 49.6 percent of the board's vote.

9. A landmark court decision on voting power involved the Nassau County, New York, Board of Supervisors. In 1964, the board had six members. The number of votes given to each was 31, 31, 21, 28, 2, and 2.

a. Determine the power index for each member.

b. The board was composed of representatives of five municipalities with the populations shown in the following table.

Hempstead	728,625
North Hempstead	213,225
Oyster Bay	285,545
Glen Cove	22,752
Long Beach	25,654

The members with 31 votes both represented Hempstead. The others each represented the municipality listed in the same order as in the table. Compare the power indices of the municipalities with their populations.

10. A *minimal winning coalition* is one in which all the voters are essential.

a. Give an example of a weighted voting situation with a winning coalition for which at least one but not all of the voters is essential. Identify the minimal winning coalitions in this situation.

b. Would defining a voter's power index as the number of minimal winning coalitions be equivalent to the definition used in this lesson? Explain.

Computer Explorations

11. Use the weighted voting program that accompanies this book to experiment with different weighted voting systems when there are three voters. Change the number of votes given to each voter and the number of votes required to pass an issue. How many different power distributions are possible? Do the same for weighted voting systems with four voters.

EU Nations Tackle Tough Issues

Noordwijk, the Netherlands, April 15, 1997

At a meeting behind closed doors European Union member states started serious negotiations on the most divisive institutional issues which the Intergovernmental Conference on the revision of the Maastricht Treaty so far avoided. The IGC ministerial session showed how far apart member states are on some of these issues.

The weighting of votes in the EU Council is a delicate issue which will certainly be settled at the last minute. The main problem is that, at present, a decision taken by a qualified majority in the Council must be backed by member states representing 60% of the EU population. The fear is that, in an enlarged Union, it might be possible for decisions to be taken by a qualified majority which in fact does not represent the majority of the EU population. Bigger member states also fear that they might be outvoted (but smaller states, like Luxemburg, tell them that, in fact, there have never been "coalitions" of "small" countries against "large" member states). In Noordwijk, Italy suggested to give Germany, Britain, France and Italy two more votes in the Council (they have ten each now) and one more to Spain (which would have nine instead of eight), in order to appease these fears.

Projects

12. The Security Council of the United Nations is composed of five permanent members and ten others who are elected to two-year terms. For a measure to pass, it must receive at least nine votes that include all five of the permanent members. Determine a power index for a permanent member and for a temporary member. (Assume that all members are present and voting.)

13. Research and report on other power indices. What, for example, is the Shapley-Shubik power index?

14. The president of the United States is chosen by the electoral college. What does this system do to the power of voters in different states in selecting the president? Research the matter and report on the relative power of voters in different states.

15. Research and report on court decisions about weighted voting.

Proportional Representation

Democracies are founded on the principle that all people should have representation in government. Most democratic countries have minority populations who feel they should be represented by one of their own members, and courts have agreed.

Unfortunately, ensuring minority representation in a legislative body like the U.S. House of Representatives is not always easy. If, for example, a state has five representatives in the U.S. House and a minority is 40% of the state's population, it seems reasonable that the minority should hold $0.4 \times 5 = 2$ of the seats. However, depending on how the boundaries of the state's five congressional districts are drawn, the minority might hold no seats.

Historically, ensuring minority representation in the U.S. House has been accomplished when district boundaries are redrawn after each census: in states with a significant minority population, some districts are established in which the minority has over half the population. Unfortunately, this practice sometimes has produced districts with a shape so unusual that courts have declared the districts unconstitutional.

How, then, does a democracy provide fair representation in government? Many democracies use an election procedure called **proportional representation**. Although there are several proportional representation systems in use, they all have a common goal: to ensure that minorities and/or political parties have representation in government proportionate to their numbers in the general population.

One form of proportional representation is achieved through a practice called **cumulative voting**. In this system, several representatives are elected from a single district. If, for example, the district has three representatives, each voter has three votes. The voter can split the three votes in any way, and can even cast more than one vote for a single candidate. Cumulative voting is used in some local elections in the United States.

Another system, the **party-list system**, is used in many European countries. In this system, each party has a list of candidates on the ballot. Each voter votes for one of the parties. When the election is over, the party receives a number of seats proportionate to the vote it received. The seats are usually assigned to the names on the party's ballot by taking the names in order from the top of the ballot until the correct number is obtained.

The **mixed member system** has voters vote for a party and a candidate. A portion of the seats are assigned to candidates and another portion to parties. All individual winning candidates receive seats. The remaining seats are awarded to members of parties that do not have a number of individual seats proportionate to the vote they received. In 1994, New Zealand voters abandoned a plurality system like the one currently used in the United States in favor of the mixed member system, which is also used in several European countries.

In the **preference vote system**, voters rank the candidates. A threshold is established, and all candidates with a vote total over the threshold are elected. Remaining seats are distributed by conducting a form of instant runoff among the remaining candidates.

Voting System Violates Law, Court Rules

Washington Post, Saturday, September 17, 1994

Richmond, Sept. 16—A federal appeals court agreed today that the election system used in a county on Maryland's Eastern Shore diluted the voting strength of black residents, but the court failed to endorse the proposed remedy.

Worcester County, in which Ocean City is located, had appealed a lower court order requiring it to use a "cumulative voting" system to elect county commissioners.

Under cumulative voting, all five of the county's commissioners would be elected at large and each voter would have five votes. A voter could cast all five votes for one candidate or split them in any way.

The system, used in only a few places across the nation as a way of increasing minority representation, had been proposed by the plaintiffs in a voting rights lawsuit against the county.

Chapter 1 Review

1. Write a summary of what you think are the important points of this chapter.

2. Consider the following set of preferences.

A	B	E	D
C	D	C	E
D	C	D	C
E	A	A	A
B	E	B	B
20	22	12	9

a. Determine the winner using a 5-4-3-2-1 Borda count.
b. Determine the plurality winner.
c. Determine the runoff winner.
d. Determine the sequential runoff winner.
e. Determine the Condorcet winner.
f. Suppose that this election is conducted by the approval method and all the voters decide to approve of the first two choices on their preference schedules. Determine the approval winner.

3. Complete the following table for the recurrence relation $B_n = 2B_{n-1} + n$.

n	B_n
1	3
2	$2(3) + 2 = 8$
3	
4	
5	

4. In this chapter, you have encountered many paradoxes involving group-ranking methods.

a. One of the most amazing paradoxes occurs when a winning choice becomes a loser when its standing actually improves. In which group-ranking method(s) can this occur?

b. Discuss at least one other paradox that occurs with group-ranking methods.

5. In the 1912 presidential election, polls showed that the preferences of voters were as follows.

Wilson	Roosevelt	Taft
Roosevelt	Taft	Roosevelt
Taft	Wilson	Wilson
45%	30%	25%

a. Who won the election? Was he a majority winner?

b. How did the majority of voters feel about the winner?

c. How might one of the groups of voters have changed the results of the election by voting insincerely?

d. Discuss who might have won the election if a different method had been used.

6. Your class is ranking soft drinks and someone suggests that the names of the soft drinks be placed in a hat and the group ranking be determined by drawing them from the hat. Which of Arrow's conditions does this method violate?

7. State Arrow's theorem. In other words, what did Arrow prove?

8. Can the point system used to do a Borda count affect the ranking (for example, a 5-3-2-1 system instead of a 4-3-2-1 system)? Construct an example to support your answer.

9. The 1992 presidential election was unusual because of a strong third-party candidate. In that election Bill Clinton received 43% of the popular vote, George Bush 38%, and Ross Perot 19%.

Steven Brams and Samuel Merrill III used polling results to estimate the percentage of those voting for one candidate who also approved of another.

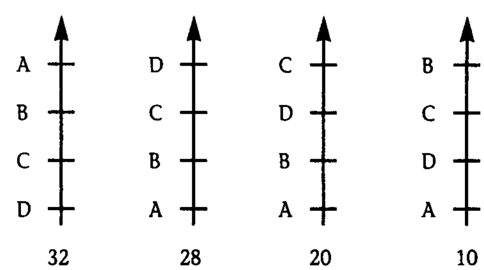
Approximately 15% of Clinton voters approved of Bush and approximately 30% approved of Perot.

Approximately 20% of Bush voters approved of Clinton and approximately 20% approved of Perot.

Approximately 35% of Perot voters approved of Clinton and approximately 30% approved of Bush.

a. Estimate the percentage of approval votes each candidate would have received if approval voting had been used in the election.
b. Find the total of the three percentages you gave as answers in part a. Explain why the total is not 100%.

10. Choose an election method from those you have studied in this chapter that you think best to use to determine the winner for the following preferences. Explain why you think your choice of method is best.



11. Consider a situation in which voters A, B, C, and D have 4, 3, 3, and 2 votes, respectively, and 7 votes are needed to pass an issue.
a. List all winning coalitions and their vote totals.
b. Find the power index for each voter.
c. Do the power indices reflect the distribution of votes? Explain.
d. Suppose the number of votes necessary to pass an issue is increased from 7 to 8. How does this change the power indices of the voters?

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