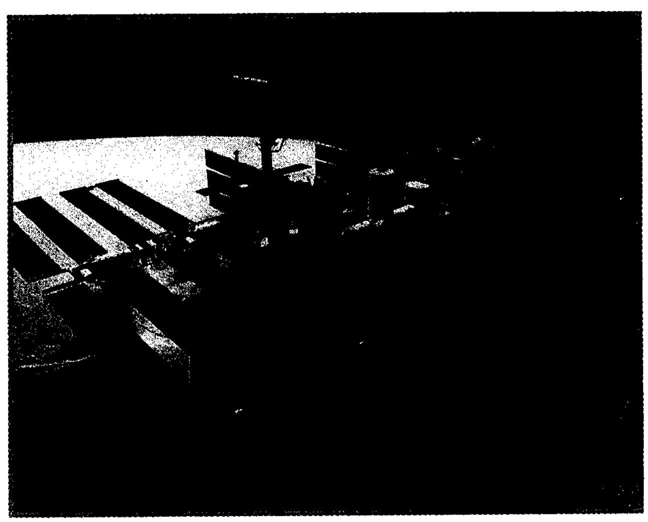
CHAPTER

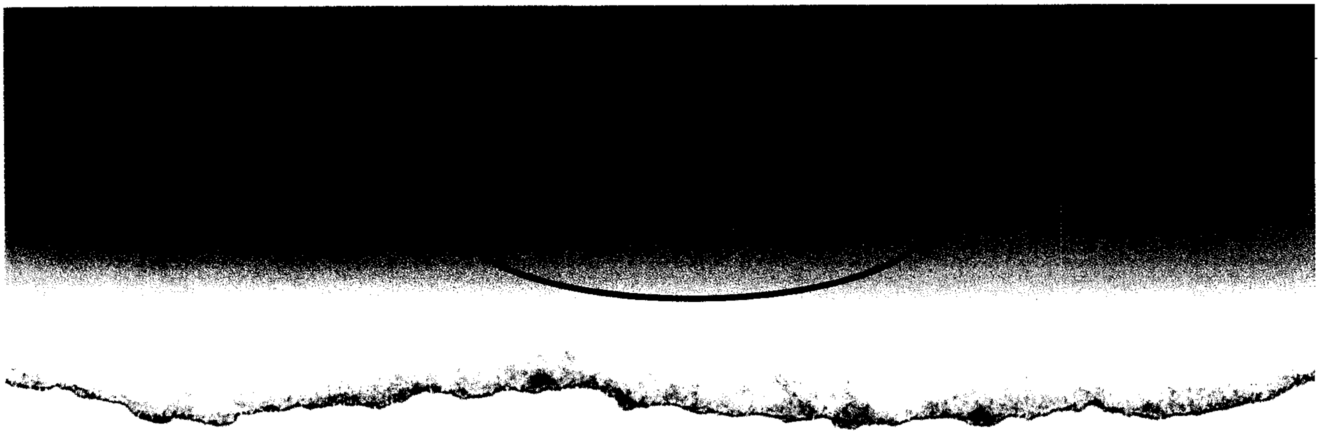
Fair Division

2

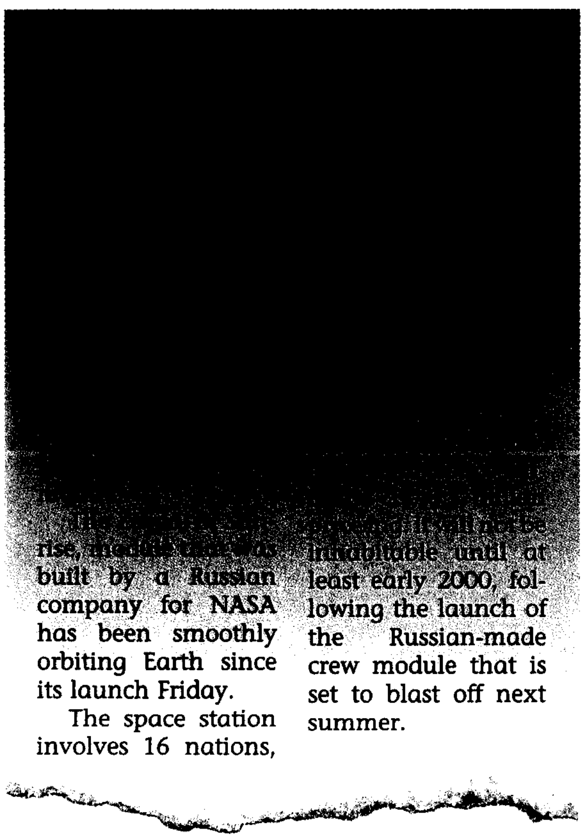
Whether the parties involved are individuals, organizations, communities, states, or nations, a joint endeavor raises questions of fairness. Massive undertakings like the international space station raise many fairness issues. For example, how will time aboard the station be divided among the nations that are paying for it? What is a fair way to divide the cost of building and maintaining the station among the nations involved?

Fair division questions arise in the simplest of situations. For example, you have no doubt experienced a feeling of unfairness when, as a child, someone else received a piece of cake or portion of ice cream you felt was better than yours. How can a portion of food be divided fairly among two or more children? Is the meaning of fairness when food is divided among children different from the meaning of fairness when an estate is divided among heirs or when seats in Congress are divided among states? Are the methods that are commonly used to divide food, estates, and legislatures necessarily the fairest methods? Discrete mathematics plays an important role in answering these questions.





A Fair Division Activity



[illegible] rise, module that was built by a Russian company for NASA has been smoothly orbiting Earth since its launch Friday.

The space station involves 16 nations, [illegible] It will not be inhabitable until at least early 2000, following the launch of the Russian-made crew module that is set to blast off next summer.

There are many ciorcumstances in which the division of an object in a fair way is important to those involved. Three of the most common are the division of food among children, a house in an estate among heirs, and the seats in a governmental body among districts. Each has characteristics that make it different from the others.

In this lesson, you will consider an example of each of these three situations and propose a solution of your own. As is the case in election theory, fair division is an area of discrete mathematics in which important problems can be understood and solved without a lot of background knowledge.

Explore This

Below is a set of three fair division problems. You will find it convenient to discuss one or more of them with a few other people, as time permits. The following is a description of one way to divide the problems among small groups in your class. At the direction of your instructor, divide your class into groups of three people. Write the numbers 1, 2, or 3 on each of several slips of paper. Have someone from each group draw one of the slips from a bag or box. Each group should consider the fair division problem listed below that corresponds to the number drawn by the group. Allow about 15 minutes for each group to discuss the problem.

After all groups have finished their discussions, a spokesperson for each group should present the group's decision to the class. Each group that discussed Problem 1 should report first, and so forth.

In your notebook, record the method used by each group. You will need the record for this lesson's exercises.

1. Martha and Ray want to divide the last piece of the cake that their mother baked yesterday. Propose a method of dividing the piece of cake that will seem fair to both Martha and Ray.

2. Juan and Mary are the only heirs to their mother's estate. The only object of significant value is the house in which they were raised. Propose a method of resolving the issue of the disposition of the house that will seem fair to both Juan and Mary.

3. The sophomore, junior, and senior classes of Central High School have 333, 288, and 279 members, respectively. The school's student council is composed of 20 members divided among the three classes. Determine a fair number of seats on the council for each class.

If your class is typical, the class members may not have reached consensus on the best way to solve each of the three problems. In this lesson's exercises you will consider some of the important fairness issues in each of the three.

Exercises

1. a. Did any groups resolve Problem 1 by relying on the mother's authority? In what way?

b. Does the resolution of such a problem by the mother or other authority figure always produce a solution that seems fair to both children? Explain.

c. Cite at least two examples of situations in which fair division problems are resolved by an authority.

2. a. Did any of the groups use a random event such as a coin flip to resolve Problem 1? In what way?
b. Does the use of randomness in such a problem always produce a solution that seems fair to both children? Explain.
c. Cite at least two examples of situations in which randomness is used to resolve an issue.

3. a. Did any of the groups use a means of measuring the piece of cake to resolve Problem 1? In what way?
b. Does the use of measurement in such problems always produce a solution that seems fair to both children? Explain.
c. Give an example of a situation in which measurement is likely to result in an agreeable solution to a fair division problem.

4. A common way of resolving Problem 1 is to have one of the children cut the cake into two pieces and to have the other choose first.
a. If Martha cuts the cake. How will she feel about the two pieces?
b. If Ray gets to choose one of the pieces that Martha cut, how will he feel about the two pieces?
c. If you were one of the participants in this scheme, would you rather be the cutter or the chooser? Why?

5. Write a description of what you consider to be desirable results of a process that fairly divides a cake among any number of people.

6. Did any of the groups resolve Problem 2 by selling the house and dividing the cash? Why might the results of such a process be unsatisfactory to one or more of the heirs?

7. Did any of the groups use a method that considers the possibility that the heirs might not agree on the value of the house? In what way?

8. Suppose that Juan thinks the house is worth $100,000 and Mary feels it is worth $120,000.
a. Who do you think should receive the house? Explain.
b. How might the person who doesn't get the house be compensated?

9. Write a description of what you consider to be desirable results of a process that fairly divides a house among several heirs.

10. How might the possibility of lying about the value of the house affect the result of a division process?

11. Did all the groups that discussed Problem 3 divide the seats among the classes in the same way? If not, describe the differences.

12. If some of the groups that discussed Problem 3 obtained different results, which of the methods do you think is the fairest? If all groups produced the same result, do you agree that the result is fair? Why or why not?

13. Write a description of what you consider to be desirable results of a process that fairly divides the seats in a student council among a school's classes.

14. Summarize the similarities and differences in the meaning of fairness in this lesson's three problems. For each of the three problems, explain why you think it is or is not possible to achieve fairness.

Projects

15. Pick a situation in which individuals or groups have developed a procedure for settling fair division problems other than the division of food, an estate, or legislative seats. Report on the method used. Compare it with methods developed in this lesson and later in this chapter.

Estate Division

A fair division problem can be either *discrete* or *continuous*. The problem of dividing a house among heirs and that of dividing a student council among classes are examples of the discrete case. Discrete division occurs whenever the objects of the division cannot be meaningfully separated into pieces. Dividing a cake is an example of the continuous case because the cake can be divided into any number of pieces.

This lesson considers fair division of an estate among heirs. In your discussions in Lesson 2.1, you may have decided that fairness is difficult to define in some situations because different people place different values on the same object. However, it is sometimes possible to use such differences of opinion to the advantage of all those involved. The following estate division algorithm produces an appealing paradox: Each of the heirs receives a share that is larger than he or she thinks is fair.

The Three Stooges. Left to right: Moe, Curly, and Larry.

An Algorithm for Dividing an Estate

1. Each heir submits a bid for each item in the estate. (Bids are not made on cash in the estate because it can be divided equally without controversy.)
2. A fair share is determined for each heir by finding the sum of his or her bids and dividing this sum by the number of heirs.
3. Each item in the estate is given to the heir who bid the highest on that item.
4. Each heir is given an amount of cash from the estate that is equal to his or her fair share (from step 2) less the amount the heir bid on the objects he or she received. If this amount is negative, the heir pays that amount into the estate.
5. The remaining cash in the estate is divided equally among the heirs.

The first four steps of the algorithm give each heir goods and cash whose total value equals what the heir feels is a fair share of the estate. The extra cash awarded in the fifth step is a bonus.

Whenever you encounter a new algorithm, a sample application of the algorithm is helpful. After considering the following example, you will have the opportunity to apply the algorithm to several different situations in this lesson's exercises.

[illegible] volving the heirs, which stalled film and merchandising deals worth millions of dollars a year. The legal challenges began in July 1993 when Fine's four grandchildren and DeRita's widow claimed the heirs of [illegible] over all rights to the characters to a company controlled by the heirs of all three. The Howard heirs, consisting of his daughter, Joan Maurer, and grandson Jeffrey Scott, subsequently declared bankruptcy.

Estate Division Example

Amanda, Brian, and Charlene are heirs to an estate that includes a house, a boat, a car, and $150,000 in cash.

Step 1

Each heir submits bids for the house, boat, and car. The bids are summarized in the following table, or matrix.

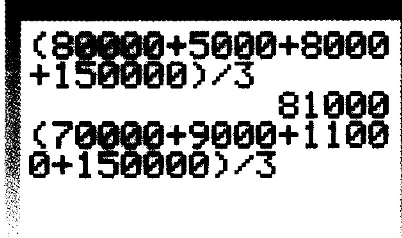
	House	Boat	Car
Amanda	$80,000	$5,000	$8,000
Brian	$70,000	$9,000	$11,000
Charlene	$76,000	$7,000	$13,000

For example, the entries in Amanda's row indicate the value to Amanda of each item in the estate.

Step 2

A fair share is determined for each heir.

Amanda: ($80,000 + $5,000 + $8,000 + $150,000) ÷ 3 = $81,000.
Brian: ($70,000 + $9,000 + $11,000 + $150,000) ÷ 3 = $80,000.
Charlene: ($76,000 + $7,000 + $13,000 + $150,000) ÷ 3 = $82,000.



new digits when necessary. Delete and Insert features should be used when digits must be eliminated or introduced.

Step 3

The house is given to Amanda, the boat to Brian, and the car to Charlene.

Step 4

Cash equal to the difference between the fair share and the value of the awarded items is given to each heir.

Amanda: \$81,000 − \$80,000 = \$1,000.

Brian: \$80,000 − \$9,000 = \$71,000.

Charlene: \$82,000 − \$13,000 = \$69,000.

Step 5

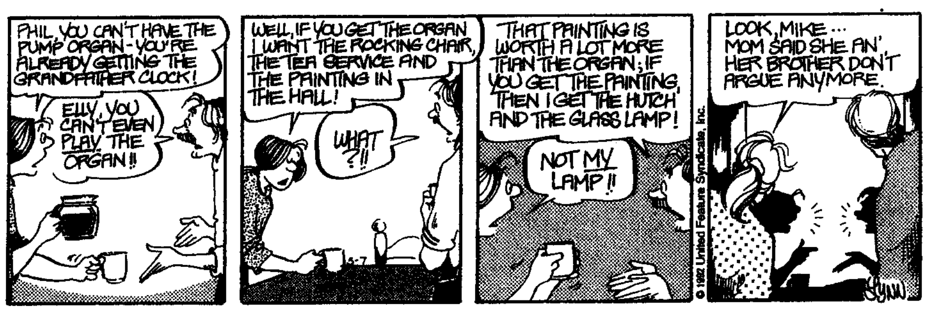
The cash given to the heirs totals \$141,000, which leaves \$150,000 − \$141,000 = \$9,000 cash in the estate. Each heir receives a bonus of \$9,000 ÷ 3 = \$3,000.

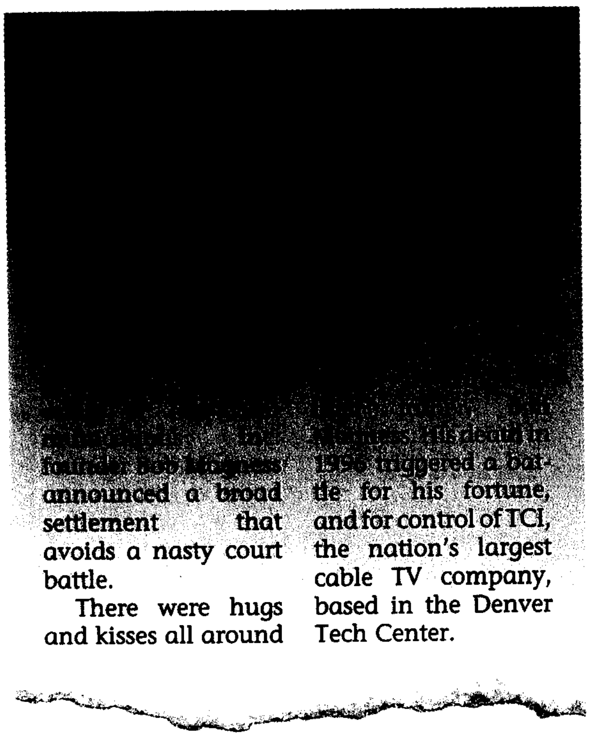
The results of this example can be summarized in a matrix:

	Amanda	Brian	Charlene
Total of bids and cash	\$243,000	\$240,000	\$246,000
Fair share	\$81,000	\$80,000	\$82,000
Items received	House	Boat	Car
Value of items received	\$80,000	\$9,000	\$13,000
Initial cash received	\$1,000	\$71,000	\$69,000
Share of remaining cash	\$3,000	\$3,000	\$3,000

For each heir, totaling the last three rows of the matrix gives the value the heir attaches to the items and cash received. For example, Amanda

For Better or For Worse® **by Lynn Johnston**




announced a broad settlement that avoids a nasty court battle.

There were hugs and kisses all around

tle for his fortune, and for control of TCI, the nation's largest cable TV company, based in the Denver Tech Center.

feels the value of her share of the estate is $80,000 + $1,000 + $3,000 = $84,000, which is more than the $81,000 that Amanda feels is a fair share.

The final settlements for each heir are:

Amanda: the house and $4,000

Brian: the boat and $74,000

Charlene: the car and $72,000

Exercises

1. The application of any fair division algorithm requires certain assumptions, or *axioms*. For example, the success of the estate division algorithm requires that each heir be capable of placing a value on each object in the estate. If any heir considers an object priceless or is otherwise incapable of placing a dollar value on an object, the algorithm fails. Give at least one other axiom that you think is necessary for the success of this algorithm.

2. Garfield and Marmaduke are heirs to an estate that contains only a summer cottage. Garfield bids $70,000, and Marmaduke bids $60,000.

a. What does Garfield feel is a fair share? Marmaduke?

b. What is the difference between Garfield's fair share and Garfield's bid for the cottage?

c. Because the value Garfield assigned to the cottage is more than Garfield's fair share, Garfield must pay cash into the estate. How much cash must Garfield pay?

d. Marmaduke is given an amount of cash from Garfield's payment equal to Marmaduke's fair share. How much does Marmaduke receive? If the remaining cash is divided equally, what will be the final value of Marmaduke's settlement? Of Garfield's?

e. Garfield must borrow money in order to pay into the estate, and the interest on this loan is $2,000. Do you think this should be considered when arriving at a settlement? If so, suggest how the settlement should be revised.

f. If the division between Garfield and Marmaduke were settled by another algorithm that is frequently used to divide estates, Marmaduke would be given half of Garfield's bid. Compare the final settlements for Garfield and Marmaduke by this method with the settlements in part d. Which result do you think is fairest? Explain.

3. Amy, Bart, and Carol are heirs to an estate that consists of a valuable painting, a motorcycle, a World Series ticket, and $5,000 in cash. They submit the bids shown in the following matrix.

	Painting	Motorcycle	Ticket
Amy	$2,000	$4,000	$500
Bart	$5,000	$2,000	$100
Carol	$3,000	$3,000	$300

a. Use the algorithm of this lesson to divide the estate among the heirs. For each heir, state the fair share, the items received, the amount of cash, and the final settlement. Summarize your results in a matrix.

A helpful hint: It is relatively easy to lose track of the estate's cash as payments are made into and out of the estate. Errors can be avoided by tracking the cash with a table designed for that purpose:

Cash in the estate	$5,000
Received from Amy	____
Received from Bart	____
Paid to Carol	____
Cash remaining	____

b. It is common for one or more heirs to pay into an estate. This lesson's algorithm fails if an heir who must pay into the estate cannot do so. Suggest a way the algorithm could be modified to account for situations in which one or more heirs cannot raise the cash necessary to complete the division.

4. Suppose that in the division of Exercise 3, Amy had received previous financial support from the estate in the form of a loan to pay college tuition. A will states that she is to receive only 20% of the estate, whereas Bart and Carol are to receive 40% each. Adapt the algorithm of this lesson to this situation and describe a fair division of the estate.

5. If two heirs submit an identical highest bid for an item, how would you resolve the tie?

6. Alan, Betty, and Carl are heirs to an estate and have submitted the bids shown in the following table.

	House	Boat	Car
Alan	\$90,000	\$4,000	\$10,000
Betty	\$95,000	\$5,000	\$8,000
Carl	\$92,000	\$4,000	\$9,000

The awarding of items in the estate can be indicated in a matrix, as shown below.

	Alan	Betty	Carl
House	0	1	0
Boat	0	1	0
Car	1	0	0

The entries in Alan's column indicate the items that he received. For example, the 1 in Alan's column and the car's row indicates that Alan received the car. Each of the other two entries in Alan's column is a 0; this indicates that Alan received neither the house nor the boat.

A new matrix can be computed by writing the second matrix beside the first, as shown below.

$$\begin{bmatrix} \$90{,}000 & \$4{,}000 & \$10{,}000 \\ \$95{,}000 & \$5{,}000 & \$8{,}000 \\ \$92{,}000 & \$4{,}000 & \$9{,}000 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The new matrix is computed by multiplying each entry in the first row of the first matrix by the corresponding entry in the first column of the second matrix and finding the sum of these products:

$$\$90{,}000(0) + \$4{,}000(0) + \$10{,}000(1) = \$10{,}000.$$

Because the result, $10,000, was obtained from the first row of the first matrix and the first column of the second matrix, it is written in the first row and the first column of the new matrix.

$$\begin{bmatrix} \$10{,}000 & \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

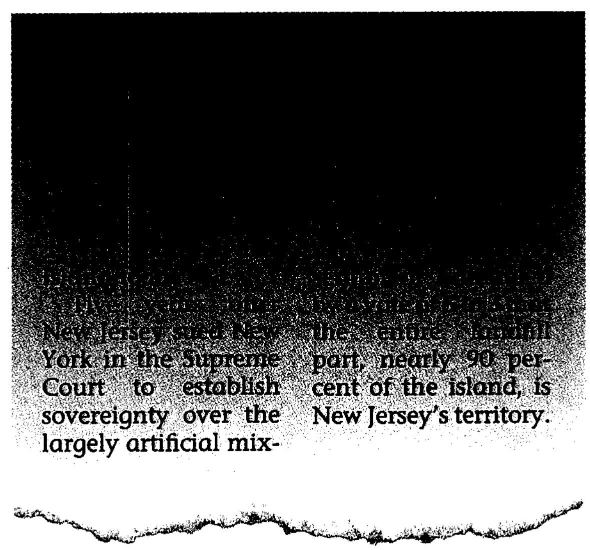
The entry for the first row and the second column of the new matrix is found by performing a similar calculation with the first row of the first matrix and the second column of the second matrix:

$$\$90{,}000(1) + \$4{,}000(1) + \$10{,}000(0) = \$94{,}000.$$

a. Calculate the remaining entries of the new matrix.
b. The $10,000 in the first row and the first column of the new matrix can be interpreted as the value to Alan of the items received by Alan. Write an interpretation of the number in the first row and the second column of the new matrix.
c. Write an interpretation of the number in the second row and the second column of the new matrix.

7. Could the estate division algorithm of this lesson encourage insincerity by any of the heirs? Explain.

8. In 1998, the U.S. Supreme Court settled a dispute between New York and New Jersey over control of Ellis Island by dividing the island between the two states. Is the problem of how to divide an island among two or more parties a continuous or discrete problem? Explain.



9. Two friends have decided to share an apartment in order to obtain a nicer apartment than either could afford individually. They choose a two-bedroom apartment that rents for $900 monthly, including utilities. One bedroom is larger and sunnier than the other. Propose a procedure for deciding which of the friends gets the nicer bedroom.

Computer/Calculator Exploration

10. It can be instructive to examine the results of an estate division when one or more of the bids is changed. However, it is tedious to redo all the calculations several times over. Fortunately, this lesson's estate division algorithm can be implemented on a computer spreadsheet, which simplifies changing values and inspecting the results. Use a computer spreadsheet to perform this lesson's estate division algorithm. A sample output and the formulas that generated it are shown below. In this case, the results are those of Exercise 3. Once your spreadsheet is complete, use it to answer the questions that follow.

	A	B	C	D	E	F
1		Estate Division Spreadsheet				
2						
3		Amy	Bart	Carol		
4	Painting	2000.00	5000.00	3000.00		
5	Motorcycle	4000.00	2000.00	3000.00		
6	Ticket	500.00	100.00	300.00		
7						
8		Amy	Bart	Carol		Cash:
9	Bid total	11500.00	12100.00	11300.00		5000.00
10	Share	0.333333	0.333333	0.333333		666.67
11	Fair share	3833.33	4033.33	3766.67		966.67
12	Object value	4500.00	5000.00	0.00		−3766.67
13	Cash received	−666.67	−966.67	3766.67		
14	Extra cash	955.56	955.56	955.56		2866.67
15						
16	Final total	4788.89	4988.89	4722.23		
17						

	A	B	C	D	E	F
1		Estate Division Spreadsheet				
2						
3		Amy	Bart	Carol		
4	Painting	2000	5000	3000		
5	Motorcycle	4000	2000	3000		
6	Ticket	500	100	300		
7						
8		Amy	Bart	Carol		Cash:
9	Bid total	=SUM(B4:B6)+$F$9	=SUM(C4:C6)+$F$9	=SUM(D4:D6)+$F$9		5000
10	Share	=1/3	=1/3	=1/3		=−B13
11	Fair share	=B9*B10	=C9*C10	=D9*D10		=−C13
12	Object value	=B5+B6	=C4	=0		=−D13
13	Cash received	=B11−B12	=C11−C12	=D11−D12		
14	Extra cash	=$F$14*B10	=$F$14*C10	=$F$14*D10		=SUM(F9:F12)
15						
16	Final total	=SUM(B12:B14)	=SUM(C12:C14)	=SUM(D12:D14)		

a. What would happen if the amount of cash in the estate were 0? Change the amount in cell F9 to 0 and see. Describe the result.
b. What would happen if Bart lied about the value he placed on the motorcycle and said he felt it was worth $5,000? Change the amount in cell C5 to 5,000 and see. (Change the cash back to 5,000 before doing this one.) Describe the result.
c. What would happen if Bart really did feel that the motorcycle was worth $5,000 but accepted a $2,000 bribe from Amy to bid $2,000? How would this collusion between Bart and Amy change the value of the final settlements for Bart and Amy?
d. Explain how to change the formulas in the spreadsheet to account for the situation in Exercise 4.

Projects

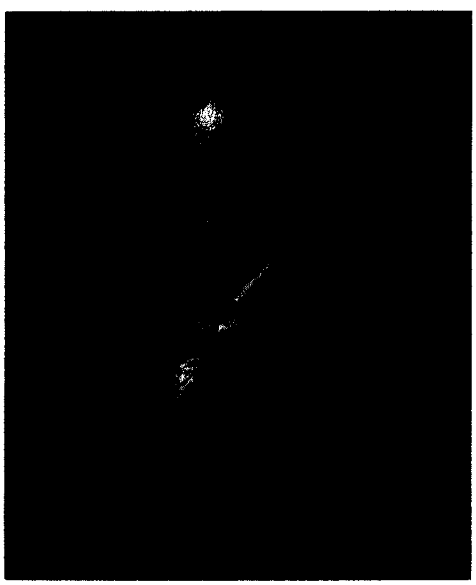
11. Matrix calculations like the multiplication shown in Exercise 6 are useful in programming computers to do tedious calculations. Research and report on the use of matrix applications in computer science.

12. Research division procedures that are used at auctions. What are Dutch and English auctions? Why do some auctions award the contract to the second-highest bidder? When are closed and open bids used?

Apportionment Algorithms

The problem of fairly dividing an estate involves discrete objects, but also involves cash. When a fair division problem is strictly discrete, the problem can be impossible to solve in a way that treats all parties fairly.

The fair allocation of discrete objects occurs in a variety of situations. For example, your school's administrators must decide a fair way to allocate teaching positions to the school's various departments and equipment such as computers to classrooms. One of the most politically charged fair distribution problems in the United States involves the apportionment of seats in the U.S. House of Representatives among the states. (The House is reapportioned every ten years after a new census is taken.)

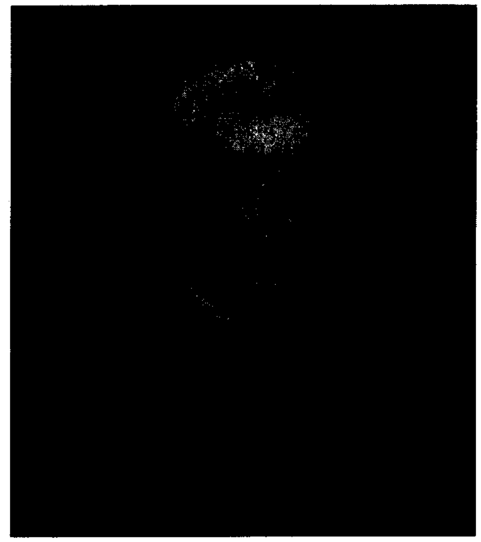


Alexander Hamilton.

Unlike estate division situations, in which individuals may not agree on the value of an object, the value of a seat in the U.S. House is not subjective. Therefore, the definition of fairness is a simple one that is mandated by the Constitution: that the seats be distributed among the states according to population.

Although the definition of fairness used to apportion seats in the U.S. House is not controversial, the method of apportionment can be. The first veto by U.S. president occurred in 1792 when George Washington rejected an apportionment bill advocated by Alexander Hamilton in favor of a method championed by Thomas Jefferson. This lesson considers the two methods of apportioning seats in a governmental body that were at the center of the Hamilton-Jefferson dispute.

Since the apportionment problem as applied to the U.S. House of Representatives involves 435 seats and 50 states, you will feel more comfortable starting with a simpler example. (Although they may seem artificial, many examples in this lesson are designed to reflect the large differences in populations among states.)



Thomas Jefferson.

Central High School has sophomore, junior, and senior classes of 464, 240, and 196 students, respectively. The 20 seats on the school's student council are divided among the classes according to population. Since there are 900 students in the school and since 900 ÷ 20 = 45, ideally each representative in the council would represent 45 students. In other words, the **ideal ratio** of students to seats is 45.

$$\text{Ideal ratio} = \frac{\text{Total population}}{\text{Number of seats}}$$

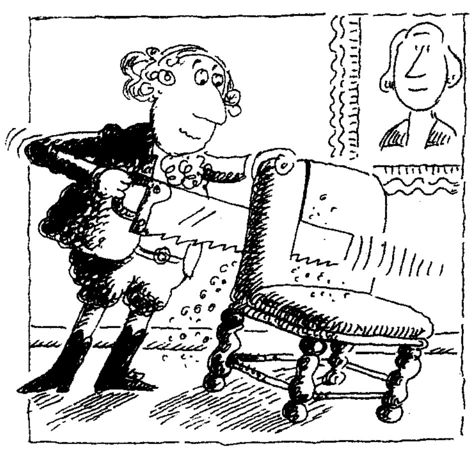
In cases of political representation, the ideal ratio is often called the *ideal district size*. If, for example, the population of the United States is 250 million, then the ideal district size is 250,000,000 ÷ 435, or about 575,000. Ideally, each member of the U.S. House would represent 575,000 people. This ideal cannot be achieved because district boundaries cannot cross state lines.

Because Central High's sophomore class has 464 members, it deserves 464 ÷ 45 = 10.31 seats. Accordingly, 10.31 is called the sophomore class **quota**. Similarly, the junior and senior quotas are 5.33 and 4.36 seats, respectively.

The Quotas	
Sophomores:	10.31
Juniors:	5.33
Seniors:	4.36

$$\text{Quota} = \frac{\text{Class Size}}{\text{Ideal Ratio}}$$

In the case of the U.S. House, a state's quota is determined by dividing its population by the ideal district size. For example, a state with 2 million people deserves 2,000,000 ÷ 575,000, or about 3.5, seats.



It isn't possible to split a seat in Central High's council and give 0.36 of it to the seniors, 0.33 of it to the juniors, and 0.31 of it to the sophomores. The school must decide a fair way to award this seat to one of the classes.

The methods favored by Hamilton and Jefferson have one thing in common: Each begins by ignoring the decimal part of each quota and assigning a number of seats equal to the whole number part of the quota. Regardless of whether the quota is 10.31 or 10.91, both Hamilton and Jefferson begin by awarding 10 seats. Ignoring the decimal part of a number in this way is called *truncating*.

The total of the truncated quotas is 10 + 5 + 4 = 19 seats. The difference in the Hamilton and Jefferson methods lies in the way they award the remaining seat.

The **Hamilton method** awards the remaining seat to the class whose quota has the largest decimal part. Since the decimal part of the senior quota, 0.36, is larger than either of the other two decimal parts, the senior class gets the extra seat. The results of the Hamilton method are summarized in the following table.

Class Size	Quota	Hamilton Apportionment
464	10.31	10
240	5.33	5
196	4.36	5

The Hamilton method seems reasonable to most people. Perhaps some members of your class proposed a similar method in Lesson 2.1. However, the Hamilton method has fallen out of favor in the United States for reasons you will consider in this lesson's exercises, after you have examined Jefferson's approach.

You might think of the **Jefferson method** as conducting a race to see whether the sophomore quota can increase to 11 or the junior quota can increase to 6 or the senior quota can increase to 5 first. Here is how it conducts this race.

Since a quota is found by dividing the class size by the ideal ratio, the quota becomes larger when the ideal ratio becomes smaller. For example, consider what happens if the ideal ratio is decreased from 45 students per seat to 40 students per seat. The results are summarized in the following table.

Class Size	Quota with Ideal Ratio of 45	Quota with Ideal Ratio of 40
464	464 ÷ 45 = 10.31	464 ÷ 40 = 11.6
240	240 ÷ 45 = 5.33	240 ÷ 40 = 6.0
196	196 ÷ 45 = 4.36	196 ÷ 40 = 4.9
Seats	10 + 5 + 4 = 19	11 + 6 + 4 = 21

For example, the sophomore class receives 10 seats when the ideal ratio is 45 and 11 seats when the ratio is 40. Therefore, there must be some ratio between 45 and 40 that causes the sophomore apportionment to be exactly 11. It can be found by dividing the sophomore class size by 11:464 ÷ 11 ≈ 42.18. The number 42.18 is called the **Jefferson adjusted ratio**.

$$\text{Jefferson adjusted ratio} = \frac{\text{Class size}}{\text{Truncated quota} + 1}$$

Similarly, the junior quota passes 6 when the ratio drops below 240 ÷ 6 = 40, and the senior quota passes 5 when the ratio drops below 196 ÷ 5 = 39.2.

Proponents of the Jefferson method argue that since the ideal ratio does not produce a complete apportionment of the seats, it should be abandoned for a new ratio, as close to the ideal as possible, that does give a complete apportionment. If the ideal ratio is gradually decreased from 45, it will reach a value at which the sophomores receive another seat before it reaches a value at which either of the other classes receives another seat, as shown in the following table.

Ideal Ratio	Sophomore Seats	Junior Seats	Senior Seats
45	10	5	4
↓			
42.18	11	5	4
↓			
40	11	6	4
↓			
39.2	11	6	5

Greater detail can be seen in the following table, in which the adjusted ratio is decreased in steps of 0.2. Note that the sophomore class quota passes the next integer before either the junior or the senior quota does.

Adjusted Ratio	Sophomore	Junior	Senior
45.00	10.31	5.33	4.36
44.80	10.36	5.36	4.38
44.60	10.40	5.38	4.39
44.40	10.45	5.41	4.41
44.20	10.50	5.43	4.43
44.00	10.55	5.45	4.45
43.80	10.59	5.48	4.47
43.60	10.64	5.50	4.50
43.40	10.69	5.53	4.52
43.20	10.74	5.56	4.54
43.00	10.79	5.58	4.56
42.80	10.84	5.61	4.58
42.60	10.89	5.63	4.60
42.40	10.94	5.66	4.62
42.20	**11.00**	5.69	4.64
42.00	11.05	5.71	4.67
41.80	11.10	5.74	4.69
41.60	11.15	5.77	4.71
41.40	11.21	5.80	4.73
41.20	11.26	5.83	4.76
41.00	11.32	5.85	4.78
40.80	11.37	5.88	4.80
40.60	11.43	5.91	4.83
40.40	11.49	5.94	4.85
40.20	11.54	5.97	4.88
40.00	11.60	**6.00**	4.90
39.80	11.66	6.03	4.92
39.60	11.72	6.06	4.95
39.40	11.78	6.09	4.97
39.20	11.84	6.12	**5.00**
39.00	11.90	6.15	5.03

The Jefferson method can be summarized in algorithmic form:

1. Divide the total population by the number of seats to obtain the ideal ratio.

2. Divide the population of each class (state, district, etc.) by the ideal ratio to obtain the class quota.
3. Assign a number of seats to each class equal to its truncated quota.
4. If the number of seats assigned matches the total number of seats to be apportioned, then stop.
5. If the number of seats assigned is smaller than the total number of seats to be apportioned, then divide the size of each class by one more than the number of seats assigned to it in step 3, to obtain an adjusted ratio.
6. Give an extra seat to the class with the largest adjusted ratio. (In other words, to the class for which the adjusted ratio is closest to the ideal ratio.)

This algorithm applies only to situations in which the total of the truncated quotas falls one short of the number of seats available. In some cases, there is more than a one-seat shortfall after truncation. This lesson's exercises consider what to do in such cases.

Exercises

1. The student council at Central High has had difficulty deciding a number of issues because of conflicts between the sophomore representatives and the representatives of the other two classes. The vote has been a 10–10 tie. The council has decided to add a seat in order to prevent frequent ties.
   a. On the basis of the data in this lesson, which class do you think should have the extra seat? Why?
   b. Find the new ideal ratio of students per seat for the 21-seat council.
   c. Use the new ratio to determine the quota for each of the three classes.
   d. Use the Hamilton method to allocate the 21 seats on the new council to the three classes.
   e. Compare the Hamilton apportionment for the 21-seat council to that of the 20-seat council and explain why the results contitute a paradox.

2. A senior council member who recently studied apportionment in the school's discrete mathematics course is unhappy over the loss of one of the senior seats and proposes the apportionment be made by a different method.

a. Find an adjusted ratio for each class as described in the Jefferson algorithm of this lesson.
b. Decrease the 21-seat ideal ratio until all 21 seats are allocated. State the number of seats given each class by the Jefferson method.
c. Compare the 21-seat Jefferson apportionment with the 20-seat Jefferson apportionment. Does the Jefferson method produce a paradox similar to the one described in Exercise 1?

3. Revise the Jefferson apportionment algorithm given in this lesson to account for situations in which more than one seat remains after truncation.

4. The paradox you observed in Exercise 1 occurs because increases in a divisor do not produce equal changes in quotients. When the size of a representative assembly increases and the total population remains the same, the ideal ratio decreases. As an example, consider two classes (states, districts, etc.) with populations of 100 and 230; an increase in the size of the council has caused the ideal ratio to decrease from 22 to 21.

a. Complete the following table and explain why it could result in the shift of a council seat from one class to the other.

Class Size	Quota with Ideal Ratio of 22	Quota with Ideal Ratio of 21
100		
230		

b. Will the paradox observed in Exercise 1 result in the loss of a seat for a small class or a large one? Why?
c. Why do you think Thomas Jefferson opposed the Hamilton method? (If you're not sure, look up the 1790 census results for Virginia, Jefferson's home state.)

5. The student council members at Central High, aware of the strange results that slight differences can make, decide to monitor the council's apportionment. At the end of the first quarter of the school year, the class numbers have changed somewhat:

Sophomores	459
Juniors	244
Seniors	197

a. Use the Hamilton method to divide the council's 21 seats among the classes.

At the end of the first semester the classes have changed again:

Sophomores	460
Juniors	274
Seniors	196

At the council's first meeting of the new semester, the members are amazed when one of the representatives of the senior class, the only class that has decreased in size, demands that the council be reapportioned.

b. Use the Hamilton method to reapportion the council.

c. Explain why the results constitute a paradox.

d. Use an analysis similar to that of Exercise 4 to explain why this type of paradox occurs. Will it have an adverse effect on small classes or large classes?

6. The cartoon on page 62 has been described as "an indiscreet attempt to apply continuous division procedure to a discrete problem." Explain what this means.

7. The U.S. population determined by the 1990 census was 248,709,873. The populations of Texas, New York, and California were 16,986,510, 17,990,455, and 29,760,021, respectively.

a. Find the quota for each of these three states.

b. Compare the quotas to the actual apportionment mentioned in the news article.

c. Do you think the apportionment treated any of these three states unfairly? Explain.

[illegible] the House of Representatives.

Texas currently boasts the third-largest House delegation with 30 seats. California is first with 52, New York second with 31.

House seats are apportioned on the basis of states' popu- [illegible] that 11 seats will shift.

The House is limited to 435 seats. Each state is guaranteed one seat and the others are then distributed on the basis of the population count mandated by the Constitution.

d. The 1990 census total for Montana was 799,065. The apportionment that followed decreased Montana's representation from seats 2 to 1. Montana went to court to challenge the apportionment method but lost its case. Do you think it would have been fairer to give Montana 1 seat or 2? Explain.

Computer/Calculator Explorations

8. Develop a spreadsheet to do the Jefferson apportionment for the three classes in this lesson's example. When finished, the spreadsheet should be similar to the following.

	A	B	C	D
1		Population	Quota	Seats
2	Sophomore	464	10.311111	10
3	Junior	240	5.333333	5
4	Senior	196	4.355556	4
5	Total	900		19
6	Seats	20		
7	Ideal ratio	45		

The values in columns C and D and in cells B5 and B7 should be calculated with formulas. The formulas in column D require the use of your spreadsheet's truncation function. On many spreadsheets, this function is abbreviated TRUNC. For example, TRUNC(C2,0) truncates the value in cell C2 so it has 0 decimal places.

When your spreadsheet is done, show how the proper apportionment can be found by changing the value in cell B7 (the ideal ratio).

Projects

9. Research and report on methods that have been used to apportion the U.S. House of Representatives and controversies that have arisen. Why has the apportionment method been changed? Why has the size of the House been changed? Did paradoxes occur with any of the methods?

10. The president of the U.S. is chosen by the electoral college. The number of electoral votes a state has is determined by the size of its congressional delegation. Thus, apportionment affects the electoral college vote. Research and report on the affect of apportionment on the election of the president. Has the apportionment method ever made a difference in whom the electoral college elected president?

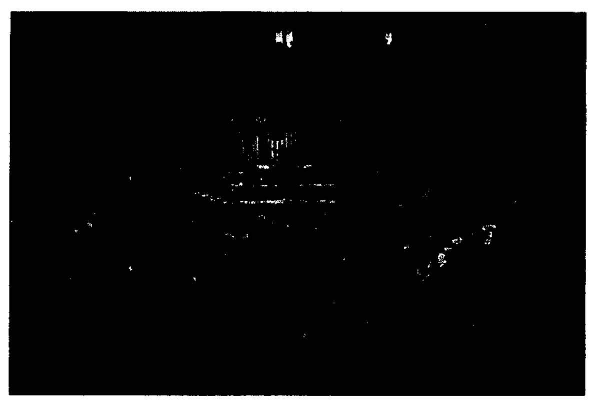
Lesson 2.4

More Apportionment Algorithms and Paradoxes

Dissatisfaction with paradoxes that sometimes occur with the Hamilton method led to its abandonment as a method of apportioning the U.S. House of Representatives. The Jefferson method was attacked by small states because it favors large states. This lesson considers alternatives to the Jefferson and Hamilton methods and some recent developments in the debate over which method of apportionment is fairest.

The Jefferson method is one of several *divisor methods* of apportionment. The term *divisor* is used because these methods determine quotas by dividing the population by an ideal ratio or an adjusted ratio. This ratio is the divisor. The Hamilton method is not a divisor method.

Two divisor methods given considerable attention today are a method named for Daniel Webster and another named for Joseph Hill, U.S. statistician. The Webster and Hill methods differ from the Jefferson method in the way that they round quotas. Recall that the



The United States House of Representatives is apportioned by the Hill method, although this has not always been the case. At various times it has been apportioned by the Jefferson method, the Hamilton method, and the Webster method.

Jefferson method truncates a quota and apportions a number of seats equal to the integer part of the quota. Quotas of 11.06 and 11.92 both receive 11 seats under the Jefferson method.

... they are underrepresented.

The Chichester School District covers Upper Chichester, Lower Chichester, Marcus Hook, and Trainer. Until now, school board members have been elected under a system in which Upper

... district presented two slightly different plans, depending on which population figures were used. In each case, part of Upper Chichester would make up one region, and in the two other regions, portions of Upper Chichester would be mixed with other municipalities.

The **Webster method** uses the rounding method with which you are familiar: A quota above or equal to 11.5 receives 12 seats, and a quota below 11.5 receives 11 seats. The number 11.5 is sometimes called the **arithmetic mean** of 11 and 12. The arithmetic mean of two numbers is the number halfway between them. It can be calculated by dividing the sum of the two numbers by 2.

The **Hill method** rounds by using the geometric mean instead of the arithmetic mean. The **geometric mean** of two numbers is the square root of their product. If the quota exceeds the geometric mean of the integers directly above and below the quota, the quota is rounded up; otherwise it is rounded down. For example, a quota between 11 and 12 must exceed $\sqrt{11 \times 12} \approx 11.4891$ to receive 12 seats (see Figure 2.1).

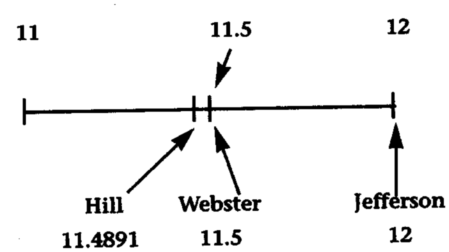


Figure 2.1 The Hill, Webster, and Jefferson roundup points for quotas between 11 and 12.

The Hill method can cause confusion because the quota for which it awards an extra seat must be calculated. For example, if the quota is 7.4903 and you are using the Webster method, you know immediately that 7 seats are awarded because the quota is slightly below 7.5. However, you do not know whether the Hill method awards 7 or 8 seats until you have calculated $\sqrt{7 \times 8} \approx 7.4833$. This calculation tells you that the Hill method awards 8 seats.

The following table summarizes the apportionment of the Central High student council example from Lesson 2.3 by the methods discussed in this and the previous lesson.

Class	Initial Apportionment				
Size	Quota	Hamilton	Jefferson	Webster	Hill
464	10.31	10	10	10	10
240	5.33	5	5	5	5
196	4.36	5	4	4	4

The Jefferson, Webster, and Hill methods all fail to assign one of the seats and so require an adjusted ratio. The following table lists the adjusted ratio necessary for each class to gain a seat by each method.

Class	Adjusted Ratio for		
Size	Jefferson	Webster	Hill
464	$464 \div 11 = 42.1818$	$464 \div 10.5 = 44.1905$	$464 \div \sqrt{10 \times 11} = 44.2407$
240	$240 \div 6 = 40.0000$	$240 \div 5.5 = 43.6364$	$240 \div \sqrt{5 \times 6} = 43.8178$
196	$195 \div 5 = 39.2000$	$196 \div 4.5 = 43.5556$	$196 \div \sqrt{4 \times 5} = 43.8269$

Recall that the adjusted Jefferson ratio for the sophomore class is found by dividing the class size by 11. The adjusted Webster ratio for the sophomore class is found by dividing the class size by 10.5. The adjusted Hill ratio for the sophomore class is found by dividing the sophomore class size by the geometric mean of 11 and 12, or 10.4881.

Each of these methods requires that the ideal ratio of 45 be decreased until it is smaller than exactly one of the adjusted ratios. To compare the methods, it can be helpful to list the adjusted ratios in decreasing order:

Jefferson: 42.1818 (sophomores), 40.0000 (juniors), 39.2000 (seniors)

Webster: 44.1905 (sophomores), 43.6364 (juniors), 43.5556 (seniors)

Hill: 44.2407 (sophomores), 43.8269 (seniors), 43.8178 (juniors)

All three methods award the extra seat to the sophomores. Note, however, that the Hill method lists the senior class second rather than third.

Because the Webster and Hill methods do not truncate quotas, they sometimes apportion too many seats rather than too few. In this lesson's exercises you will consider how to apply the Webster and Hill methods in such cases and learn of some surprising results.

Exercises

1. a. Complete the following apportionment table for the 21-seat Central High student council described in Exercise 1 of Lesson 2.3 (see page 65).

Class Size	Initial Apportionment: Quota	Hamilton	Jefferson	Webster	Hill
464	____	____	____	____	____
240	____	____	____	____	____
196	____	____	____	____	____

b. The Jefferson method distributes only 19 seats. You can use your results from Exercise 1 of Lesson 2.3 to complete the Jefferson apportionment. Give the final Jefferson apportionment.

Both the Webster and Hill methods apportion 22 seats. Therefore, the ideal ratio must be increased until one of the classes loses a seat. The sophomore class, for example, will lose a seat under the Webster method if its quota drops below 10.5. This requires an adjusted ratio of $464 \div 10.5 \approx 44.1905$. For the sophomore class to lose a seat under the Hill method, its quota must drop below $464 \div \sqrt{10 \times 11} \approx 44.2407$.

c. Complete the following table of adjusted ratios for the Webster and Hill methods.

Class Size	Adjusted Ratio For: Webster	Hill
464	$464 \div 10.5 = 44.1905$	$464 \div \sqrt{10 \times 11} = 44.2407$
240	____	____
196	____	____

d. List the adjusted ratios for the Webster method in increasing order. The ideal ratio must be increased until it passes the first ratio in your list. This class loses one seat. Give the final Webster apportionment.
e. List the adjusted ratios for the Hill method in increasing order. The ideal ratio must be increased until it passes the first number in your list. This class loses one seat. Give the final Hill apportionment.
f. Which apportionment methods would be favored by each class? Explain.

2. Since the number of seats assigned to a class rarely equals its quota exactly, apportionments are not completely fair. This fact has led people to ask whether one apportionment is less fair than another. One way to measure the unfairness of an apportionment is to total the discrepancies between the quota and the number of seats assigned to each class. For example, if the quota is 11.25 seats and 11 seats are apportioned, the unfairness is 0.25 seats. If 12 seats are apportioned, the unfairness is 0.75 seats.

a. Use the apportionments for the 20-seat Central High student council to measure the discrepancy for each class by means of each method. Record the results in the following table.

Class		Amount of Discrepancy			
Size	Quota	Hamilton	Jefferson	Webster	Hill
464	10.31	____	____	____	____
240	5.33	____	____	____	____
196	4.36	____	____	____	____
Total discrepancy		____	____	____	____

b. Which method has the smallest total discrepancy?
c. Do you think smallest total discrepancy is a good criterion for choosing an apportionment method? Explain.

3. Another way to measure the unfairness of an apportionment is to compare the representation of two classes (states, districts, etc.) as a percentage. For example, if a class with 250 students has 5 seats, then each seat represents $250 \div 5 = 50$ people. If another class has 270 students and 6 seats, the representation is $270 \div 6 = 45$ people per seat. The representation is unfair to the first class by $50 - 45 = 5$ people per seat, which is $5 \div 50 = 0.10$, or 10% of its representation.

a. The initial Hill apportionment in Exercise 1 apportioned one seat too many. Compare the representation of the junior and senior classes if one seat is taken from the juniors. What is the unfairness percentage to the juniors?

b. Compare the representation of the junior and senior classes if one seat is taken from the seniors. What is the unfairness percentage to the seniors?

c. By this percentage measure, is it less unfair to take a seat from the juniors or to take a seat from the seniors? Which apportionment in Exercise 1 agrees with your answer?

4. None of the divisor methods is plagued by the paradoxes that caused the demise of the Hamilton method. Divisor methods, however, can cause their own peculiar problems. As an illustration, consider the case of South High School. The freshman class has 1,105 members, the sophomore class has 185, the junior class 130, and the senior class 80. The 30 members of the student council are apportioned among the classes by the Webster method.

a. What is the ideal ratio?

b. Complete the following Webster apportionment table.

Class Size	Quota	Initial Webster Apportionment
1,105	22.1	22
185	___	___
130	___	___
80	___	___

c. Because the Webster method apportions too many seats, the ideal ratio must be decreased. Calculate the adjusted ratio necessary for each class to lose a seat and enter the results in the following table.

Class Size	Adjusted Ratio
1,105	1105 ÷ 21.5 = 51.3953
185	___
130	___
80	___

d. Determine the final apportionment. Explain your method.

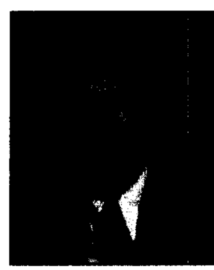
e. Explain why the freshman class would consider the final apportionment unfair.

The Webster apportionment in Exercise 3 demonstrates a **violation of quota**. This occurs whenever a class (district, state, etc.) is given a number of seats that does not equal either the integer part of its quota or one more than that. It can occur with any divisor method and is considered a flaw of divisor methods.

The following table shows how the quotas for each class change as the ideal ratio is gradually increased. The freshman quota drops much more quickly than the others do.

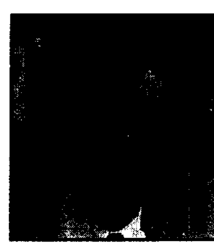
Adjusted Ratio	Freshman	Sophomore	Junior	Senior
50.00	22.10	3.70	2.60	1.60
50.20	22.01	3.69	2.59	1.59
50.40	21.92	3.67	2.58	1.59
50.60	21.84	3.66	2.57	1.58
50.80	21.75	3.64	2.56	1.57
51.00	21.67	3.63	2.55	1.57
51.20	21.58	3.61	2.54	1.56
51.40	21.50	3.60	2.53	1.56
51.60	21.41	3.59	2.52	1.55
51.80	21.33	3.57	2.51	1.54
52.00	21.25	3.56	2.50	1.54

Around 1980 Michel L. Balinski and H. Peyton Young proved an important impossibility theorem: Any apportionment method sometimes produces at least one of these undesirable results: violation of quota and the two paradoxes that occur with the Hamilton method (see Exercises 1 and 5 of Lesson 2.3).



Michel L. Balinski is a director of the Laboratoire d'Econometrie and the Ecole Polytechnique in Paris.

5. Every person in a small community belongs to exactly one of the community's four political parties. Membership is distributed as shown in the following table.



H. Peyton Young is professor of economics at Johns Hopkins University.

Party	Membership
A	561
B	200
C	100
D	139

The 20 seats in the community's council are apportioned by the Jefferson method.

a. Determine the Jefferson apportionment.
b. Parties C and D decide to join together to form a single party. Determine the new apportionment.
c. Would the apportionment that results from the merger of parties C and D have occurred with any of the other methods you have studied? Explain.

6. Compare the current quota for Upper Chichester with the one proposed by its residents in the news article on page 70. Do you feel the Upper Chichester residents are justified in their demands? How does this situation compare with weighted voting situations you studied in Lesson 1.5?

7. Compare the Hill roundoff point with the Webster roundoff point for quotas of various sizes. How, for example, do they compare for quotas between 2 and 3, between 10 and 11, between 100 and 101? Can you prove any relationship between the two?

Computer/Calculator Explorations

8. Extend the spreadsheet you made in Exercise 8 of Lesson 2.3 to include columns for the Webster and the Hill methods. You will need to use your spreadsheet's rounding function to do the Webster apportionment. For the Hill apportionment you will need to use the spreadsheet's logical functions. Logical functions on a spreadsheet are similar to those on a calculator. For example, the calculation $A(B < 5) + (A + 1)(B > 5)$ produces the value of A when $B < 5$ and the value of $A + 1$ when $B > 5$ because an equation or inequality has a value of 0 when it is false and 1 when it is true.

Projects

9. Research and report on the work of Balinski and Young. How did they prove their result? What method of apportionment did they recommend for the U.S. House of Representatives?

10. Research divisor methods for other ways in which violation of quota can occur. For example, can you find four classes for which the number of apportioned seats falls two short of the size of the council and results in awarding both seats to the same class?

11. Investigate measures of fairness such as those given in Exercises 2 and 3. For each apportionment method discussed in this and the previous lesson, which measure of fairness produces the same apportionment?

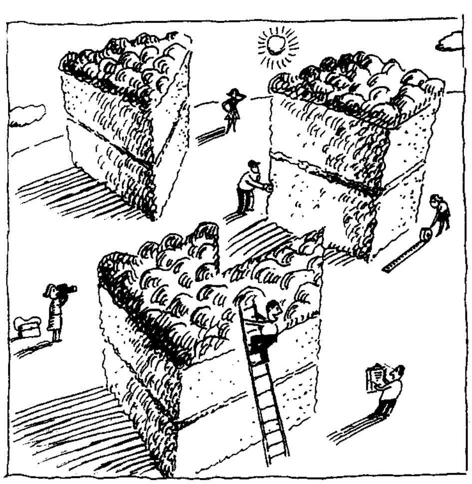
12. Research and report on the process used by the U.S. Census Bureau to apportion U.S. House seats after each census. How does the bureau implement the Hill method?

13. Research and report on apportionment methods used in other countries. Which methods are used and why?

14. Research and report on the way presidential primary results are used to apportion convention delegates among the candidates.

Fair Division Algorithms: The Continuous Case

The problem of fairly dividing a cake is in some ways similar to estate division. Like the cash in an estate, a cake can be divided in any number of ways. Moreover, the participants in a cake division situation may not agree on the value of a particular slice of cake, just as the heirs to an estate may place different values on a house. However, unlike estate division and legislative apportionment, a cake division problem is strictly continuous and involves no items such as cash and legislative seats that are of indisputable value.



The problem of fairly dividing a continuous medium is not limited to the activities of children. States, for example, must agree on how to share water resources.

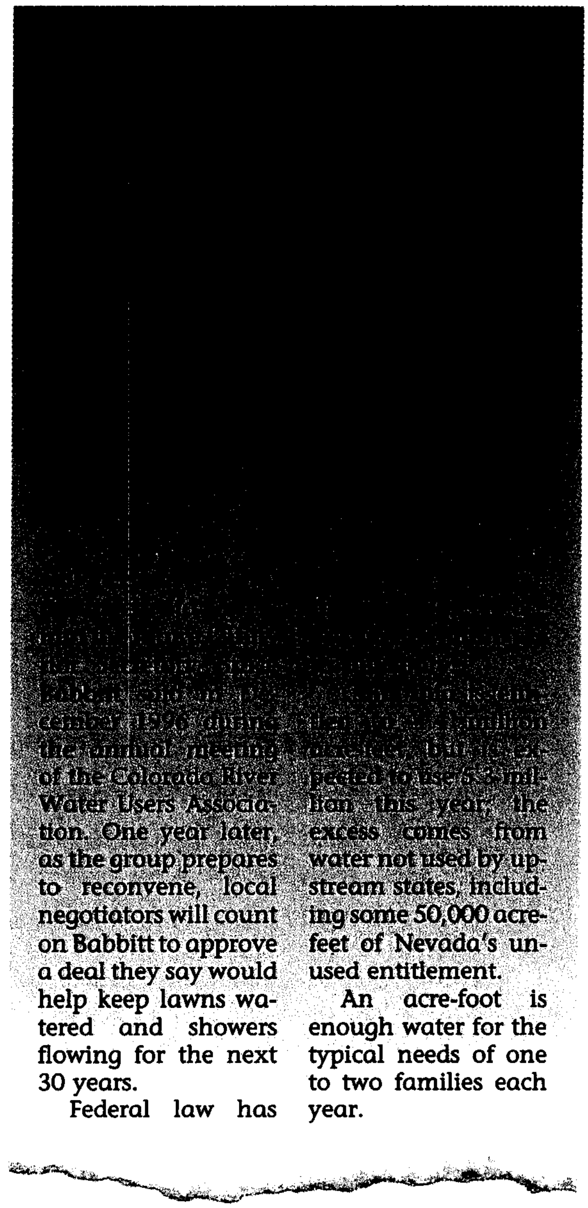
Before beginning your exploration of cake division, you may want to review the work that you and others in your class did in Lesson 2.1. In that lesson, you concluded that the resolution of a cake division problem by an outside authority may not yield a solution that is fair in the eyes of both individuals. You also concluded that the use of a random event such as a coin toss may not result in a division that is fair in the eyes of both parties. In order for two people to feel that a piece of cake has been divided fairly, each must feel that he or she received at least half the cake.

An acceptable solution to the problem of fairly dividing a piece of cake is not possible without a *definition of fairness*. Therefore, a division among n people is called fair if each person feels that he or she received at least $1/n$th of the cake (or other object).

Lesson 2.1 also showed that an appeal to a measurement scheme such as weighing may not be adequate because an individual's evaluation of a piece of cake may be based on more than just size. Cake icing, for example, could mean more to one person than to another. Along with a definition of fairness, a good solution to the cake division problem requires a few realistic *assumptions* about the participants.

1. Each individual is capable of dividing a portion of the cake into several portions that the individual feels are equal.
2. Each individual is capable of placing a value on any portion of the cake. The total of the values a person places on all portions of a cake is 1, or 100%.
3. The value that each individual places on a portion of the cake may be based on more than just the size of the portion.

The two-person cake division problem is usually solved by having one person cut the cake into two pieces and letting the other person choose one of



[illegible] cember 1996 during the annual meeting of the Colorado River Water Users Association. One year later, as the group prepares to reconvene, local negotiators will count on Babbitt to approve a deal they say would help keep lawns watered and showers flowing for the next 30 years.

Federal law has [illegible] pected to use 5.3 million this year; the excess comes from water not used by upstream states, including some 50,000 acre-feet of Nevada's unused entitlement.

An acre-foot is enough water for the typical needs of one to two families each year.

them. It is instructive to examine the roll of the above definition and assumptions in this solution.

The requirement that one person (the cutter) cut the cake into two portions that the person feels are equal is the first assumption. The second assumption ensures that the second person (the chooser) will place values on the two pieces of the cake that total 1. However, because of the third assumption, the chooser need not place the same value on both pieces. Even if the chooser feels that his or her piece is more than half the cake, the division is fair because the definition of fairness requires only that each person feel that his or her piece is at least half the cake.

No solution to the problem of fairly dividing a cake among more than two people is adequate unless it adheres to the definition of a fair cake division. For example, a fair division among three people must result in each person receiving a piece the person feels is at least one-third of the cake.

In mathematics, there is often more than one way to solve a problem. That is the case with the problem of dividing a cake fairly among three or more people. Often a solution to a problem can be based on the solution to a simpler problem. The following solution to the three-person problem is based on the two-person solution.

Call the three individuals Ann, Bart, and Carl. The solution is described in algorithmic form:

1. Ann cuts the cake into two pieces that she feels are equal.
2. Bart chooses one of the pieces; Ann gets the other.
3. Ann cuts her piece into three pieces that she considers equal; Bart does the same with his.
4. Carl chooses one of Ann's three pieces and one of Bart's.

To see that this is indeed a solution to the three-person problem, you must be satisfied that it adheres to the definition of fairness. That is, you must believe that each person has received a portion that the person feels is at least 1/3 of the cake.

Consider Ann. In step 1, Ann feels that each piece is one-half of the cake. She therefore feels that the piece she received in step 2 is half the cake. She feels that each piece she cut in step 3 is one-third of half the cake, or one-sixth of the cake. She therefore feels that she receives two-sixths, or one-third, of the cake.

Bart's case is similar except that he may feel that the portion he chose in step 2 is more than half the cake. Thus, he may feel that each piece he cut in step 3 is more than one-sixth of the cake and that his final share is more than two-sixths, or one-third.

Carl's case is quite different from either Ann's or Bart's. He may feel that the two pieces Ann cut in step 1 are unequal. He may, for example, consider one piece 0.6 of the cake and the other 0.4. Likewise, he may not feel that the cuts made in step 3 resulted in equal pieces. He could, for example, decide that the piece he valued at 0.6 was divided into pieces he values at 0.3, 0.2, and 0.1. Similarly, he could decide that the piece he valued at 0.4 was divided into pieces he values at 0.2, 0.1, and 0.1. However, because he chooses first, he will pick the largest piece from each: 0.3 from the piece he valued at 0.6 and 0.2 from the piece he valued at 0.4. Thus, he feels that the value of the portion he receives is $0.3 + 0.2 = 0.5$.

The previous example gave Carl a portion he valued at more than one-third, but it was only one example. To see that Carl will always get a portion he values as at least one-third, use the variable x to represent the value Carl places on one of the pieces cut by Ann in step 1. His value for the other piece cut by Ann must be $1 - x$. Although Carl may not feel that the piece he values at x has been divided into equal thirds, he will choose the largest of the three pieces and value it as at least one-third of x, or $\frac{1}{3}x$. Similarly, he will value the piece he chooses from the part he values as $1 - x$ as at least $\frac{1}{3}(1 - x)$. Thus, the total value of his two pieces is at least $\frac{1}{3}x + \frac{1}{3}(1 - x) = \frac{1}{3}x + \frac{1}{3} - \frac{1}{3}x = \frac{1}{3}$. This method is referred to as the **cut-and-choose method**.

This lesson's exercises consider several issues related to cake division and several ways of solving problems with three or more participants.

Exercises

1. In the division among Ann, Bart, and Carl, who will value his or her share as exactly one-third? Who might feel that he or she received more than one-third? Explain.

2. Does the division among Ann, Bart, and Carl result in three portions or three pieces? Does your answer violate the definition of fairness or any of the three assumptions?

3. For each of the four steps of the division among Ann, Bart, and Carl, state which of the three fairness assumptions is applied.

In Exercises 4 and 5, suppose Carl feels that Ann's initial division in step 1 is even, that Ann's subdivision in step 3 is also even, but that

Bart's subdivision in step 3 is not. (Give your answers as fractions or as decimals rounded to the nearest 0.1.)

4. What value will Carl place on the piece he takes from Ann? Explain.

5. Although Carl feels that the piece Bart divided is half the cake, he does not feel that Bart subdivided it equally. He could, for example, place values of 0.3, 0.1, and 0.1 or values of 0.4, 0.06, and 0.04 on the three pieces.

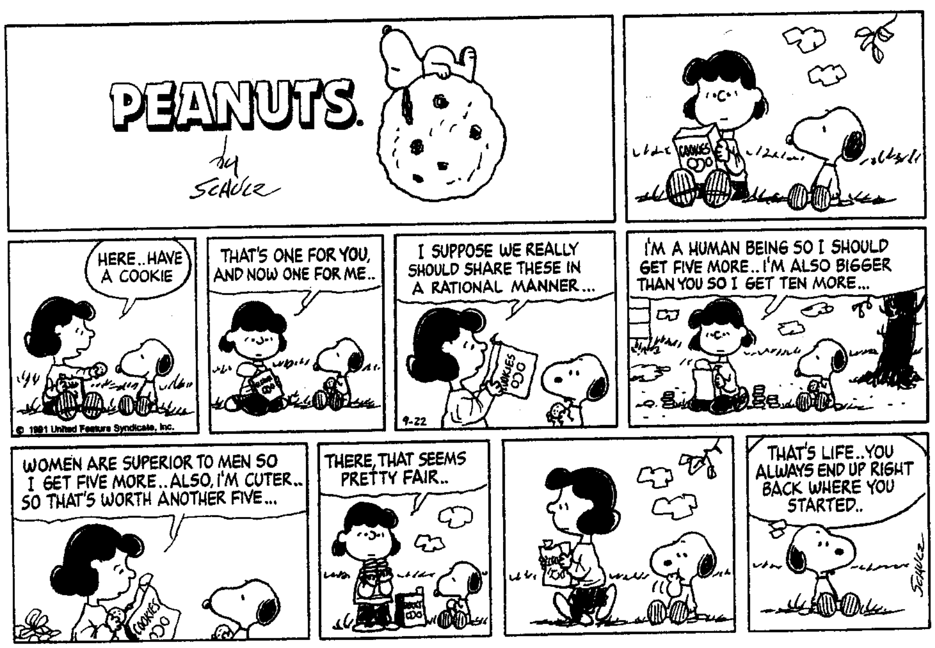
a. Explain why the largest value Carl can place on any of Bart's three subdivisions is 0.5.

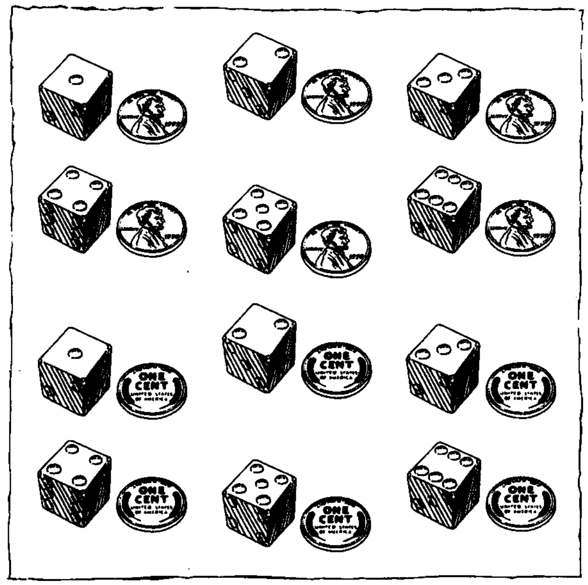
b. What is the smallest value Carl could place on the piece he takes from Bart? Explain.

c. What is the largest total value Carl could place on the two pieces he takes from Amy and Bart?

d. What is the smallest value Carl could place on the two pieces he takes from Amy and Bart?

6. Is the problem of fairly dividing cookies among three children discrete or continuous? Explain.





7. In mathematics, a fundamental principle of counting is that if there are m ways of performing one task and n ways of performing another, then there are $m \times n$ ways of performing both. For example, a tossed coin may fall in two ways, and a rolled die may fall in six ways. Together they may fall in a total of $2 \times 6 = 12$ ways.

a. If two people each have a piece of cake and each person cuts his or her piece into three pieces, show how to use the fundamental counting principle to determine the number of pieces that result.

b. If k people each have a piece of cake and each cuts his or her piece into $k + 1$ pieces, what are two equivalent expressions for the total number of pieces that result?

c. If $k + 1$ boxes each contain $k + 5$ toothpicks, what are two equivalent expressions for the total number of toothpicks?

d. Two offices are being filled in an election: mayor and governor. If there are three candidates for governor and four for mayor and conventional voting procedures are used, in how many ways can a person vote?

8. Consider the following division of a cake among three people: Arnold, Betty, and Charlie. Arnold cuts the cake into three pieces. Betty chooses one of the pieces, and Charlie chooses either of the remaining two. Arnold gets the third piece.

a. Will Arnold feel he has received at least one-third of the cake? Might he feel he has received more than one-third?

b. Will Betty feel she has received at least one-third of the cake? Might she feel she has received more than one-third?

c. Will Charlie feel he has received at least one-third of the cake? Might he feel he has received more than one-third?

9. Arnold, Betty, and Charlie decide to divide a cake in the following way: Arnold slices a piece he considers one-third of the cake. Betty inspects the piece. If she feels it is more than one-third of the cake, she must cut enough from the cake so that she feels it is one-third of the cake. The removed portion is returned to the cake. Charlie now inspects Betty's piece and has the option of doing similarly if he thinks it is still more than one-third of the cake. The piece of cake is given to the last person who cut from it.

One of the remaining two people slices a piece that he or she feels is half of what remains of the cake. The other person inspects the piece with the option of removing some of the cake if he or she feels it is more than half the remainder.

a. Will the person who receives the first piece feel that it is at least one-third of the cake? Could he or she feel it is more than one-third? Explain your answers.
b. Will the person who receives the second piece feel that it is at least one-third of the cake? Could he or she feel it is more than one-third? Explain your answers.
c. Will the person who receives the third piece feel that it is at least one-third of the cake? Could he or she feel it is more than one-third? Explain your answers.

This method is called the **inspection method.**

10. The definition of a fair cake division used in this lesson does not state that each person should feel his or her portion of the cake is at least as large as the portion received by each of the other participants.

a. Construct an example for the three-person cut-and-choose division among Ann, Bart, and Carl to show that Carl may feel one of the others got a piece bigger than his.
b. Could the definition of fair cake division result in jealously? Explain.

Computer/Calculator Exploration

11. Use the moving knife program that accompanies this book to divide a cake of any shape among yourself and two other people in your class by means of the **moving knife method**.

a. Explain why each of the people in your group feels that he or she received at least one-third of the cake.
b. Might any of the people feel jealous of the share received by another? Explain.

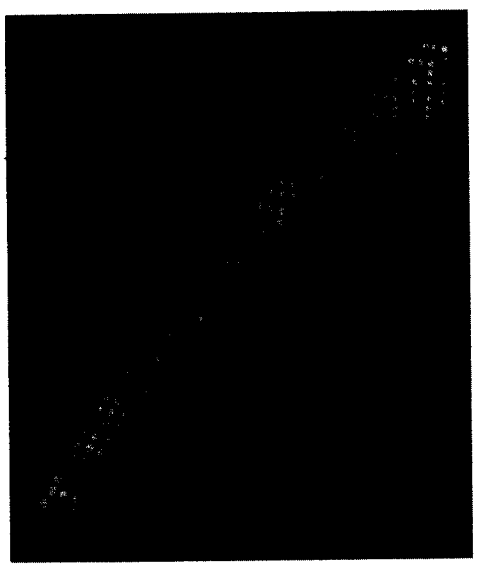
12. A procedure for fairly dividing a cake can be considered "optimal" if it fulfills some criterion. For example, a method might be called optimal if it results in exactly the same number of pieces as participants. By this criterion, are the cut-and-choose method, the inspection method, or the moving knife method optimal? Explain.

13. Could the cut-and-choose, the inspection, or the moving knife method be extended to divide a cake among four people? Explain how this could be done.

Mathematical Induction

The cut-and-choose method, the inspection method, and the moving knife method all produce a fair division of a cake among two or three people. It appears that all these methods can be extended to larger groups, but you or others in your class may not be convinced. This lesson considers a method mathematicians use to prove that a discrete process will work indefinitely. The method is called **mathematical induction**.

For example, consider the cut-and-choose method. Since mathematical induction is used to prove that a result known to work in a few cases can be extended indefinitely, it is important to review the situations for which the cut-and-choose method has been demonstrated. It was first offered as a means by which two people could fairly divide a cake. It was extended to three people by requiring that two of them first apply the two-person method. Then each of the two cut his or her piece into three pieces that he or she considered equal. The third person selected a piece from each. Therefore, the cut-and-choose method is known to work for groups of two and three people.



Before applying mathematical induction, consider how to extend the cut-and-choose method to four people, Ann, Bart, Carl, and Daiva. Begin by having Ann, Bart, and Carl apply the three-person method. Then each of them divides his or her portion into four portions that he or she considers equal. Daiva chooses one from each. To be sure this division is fair, you must agree that each of the three cutters and the chooser feel that the received share is one-fourth of the cake.

Consider one of the cutters, Ann. The three-person solution guarantees that she feels she has at least one-third of the cake. Therefore, Ann divides what she considers a third into four equal pieces. Since she retains three of these pieces, the value she attaches to her portion is at least $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$.

Daiva may not feel that the three-person solution produced three equal portions, but must feel that the total value of the three portions is 1. Suppose Daiva attaches values of p_1, p_2, and p_3 to the three portions, where $p_1 + p_2 + p_3 = 1$. Because Daiva is given first choice of a portion from each of the cutters, she will place a value of at least $\frac{1}{4}p_1 + \frac{1}{4}p_2 + \frac{1}{4}p_3 = \frac{1}{4}(p_1 + p_2 + p_3) = \frac{1}{4}(1)$ on the resulting portion. Therefore, Daiva considers the division fair.

It appears that this division can be extended to a 5-person situation, to a 6-person situation, and, in general, to an n-person situation. If, for example, a cake is to be divided fairly among 10 people, the division can be based on a 9-person division, which, in turn, is based on an 8-person division, which is based on . . . .

Mathematical induction generalizes this pattern of solutions by proving it is always possible to extend the solution to a group that is one larger than the previous. The generalization is achieved by using a variable instead of a specific number. In other words, assuming you know how to divide a cake fairly among k people, you must show that it is possible to use that knowledge to divide a cake fairly among $k + 1$ people.

The proof begins by applying the assumption that k people can fairly divide the cake. When they have finished the division, each of them divides his or her share into $k + 1$ portions that he or she feels are equal. The $(k + 1)$th person then selects one portion from each.

It is now necessary to show that each of the $k + 1$ people has a share valued as at least $\frac{1}{k+1}$.

In the case of the k cutters, the k-person solution guarantees that each has a portion valued as at least $\frac{1}{k}$ of the cake. This portion is divided into $k + 1$ equal portions, so the value of each is at least $\frac{1}{k(k+1)}$. Since k of the $k + 1$ portions are retained by the cutter, the total value attached to them is

$$k\left(\frac{1}{k(k+1)}\right) = \frac{1}{k+1}$$

Although the chooser may not feel that each of the k portions that result from the k-person solution is $\frac{1}{k}$ of the cake, he or she must feel that the total value is 1. Suppose this person assigns values of $p_1, p_2, \ldots, p_k$ to the k portions. Then $p_1 + p_2 + \cdots + p_k = 1$. Because the chooser is given first choice of a portion from each of the cutters, the chooser will place a value of at least

$$\frac{1}{k}p_1 + \frac{1}{k}p_2 + \cdots + \frac{1}{k}p_k = \frac{1}{k}(p_1 + p_2 + \cdots + p_k) = \frac{1}{k}(1)$$

on the resulting portion. Therefore, the chooser considers the division fair.

The proof is complete because it demonstrates that whenever a cake can be divided fairly among k people, it can also be divided fairly among $k + 1$ people. In other words, the cut-and-choose method can be extended indefinitely. Note that the mathematical induction argument is merely a generalization of the one given to justify the extension to four people.

Keep in mind that it is senseless to attempt to show that a process can be extended unless it is known to work in at least one instance. In the case of the cut-and-choose method, previous work had shown it applied to groups of two and three.

Verifying a Formula with Mathematical Induction

One of the most common uses of mathematical induction is to verify that a formula that describes a numerical pattern will do so indefinitely. The identification and verification of numerical patterns helps people analyze and predict trends in science, business, economics, world affairs, and many other areas. By confirming that a formula accurately describes a pattern, mathematical induction helps avoid erroneous predictions that can waste time and money.

For example, consider a situation in which Luis and Britt are investigating the number of handshakes that are made in a group of people if every person shakes hands with every other person. Luis notes that if there is only one person in the group, no handshakes are made and that if there are two people, one handshake is made. Britt draws a graph (see Figure 2.2) in which the vertices represent people, and the segments, or edges, represent

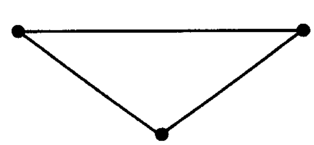


Figure 2.2 Graph representing handshakes among three people.

handshakes. She concludes that a group of three people requires a total of three handshakes.

Luis suggests organizing the data into a table.

Number of People in the Group	Number of Handshakes
1	0
2	1
3	3

Britt and Luis must now determine a formula that predicts the number of handshakes in a group from the number of people in the group. They will need mathematical induction to prove that any conjectured formula will work for a group of any size. In this lesson's exercises you will help Britt and Luis complete their work and then try some similar problems on your own.

Exercises

1. To use mathematical induction, you must be able to express numerical patterns in symbols. Some of the expressions you write in this exercise will be used in the mathematical induction proof of a formula for the handshake problem.

a. If there are three people in a group and another person joins the group, there will be four people in the group. If a person leaves the original group, there will be two. Write an expression for the number of people if there are k people in the group and another person joins. Do the same if a person leaves the group of k people.

b. Repeat part a if the original group has $k + 1$ people and if the original group has $2k$ people.

2. Draw a graph like Britt's and make a table like Luis's.

a. Add another vertex to the graph to represent a fourth person, and draw segments to represent the additional handshakes that result from the addition of the fourth person. In your table, write the total number of handshakes in a group of four people.

b. Add a fifth vertex to represent a fifth person and draw segments to represent the additional handshakes. In your table, write the total number of handshakes in a group of five people.

3. a. Suppose that there are seven people in a group and each of them has shaken hands with each of the others. If an eighth person enters the group, how many additional handshakes must be made? Explain.

b. Suppose that there are k people in a group and each of them has shaken hands with every other person. If a new person enters the group, how many additional handshakes must be made? Explain.

c. If H_n represents the number of handshakes in a group of n people, write a recurrence relation that expresses the relationship between H_n and H_{n-1}. Write a recurrence relation that expresses the relationship between H_{n+1} and H_n.

4. After studying the data, Britt wonders whether the number of handshakes in a group can be found by multiplying the number of people by the number that is 1 less than that and then dividing by 2.

a. If her guess is correct, how many handshakes would there be in a group of 10 people?

b. On the basis of Britt's guess, write an expression for the number of handshakes if there are k people in a group. Do the same for a group of $2k$ people. Do the same for a group of $k + 1$ people.

Britt's formula, if correct, is sometimes called a solution of the recurrence relation you wrote in part c of Exercise 3. One of the solution's advantages is that it allows you to determine the number of handshakes in a group without knowing the number of handshakes in a smaller group.

To prove that Britt's guess is correct, you must show that whenever the solution is known to work, it is possible to extend it to a group that is one larger. In other words, whenever the conjecture works for a group of k people, it will also work for a group of $k + 1$ people.

c. Assume that Britt's formula works for a group of k people. Write her formula for the number of handshakes in a group of k people.

d. You must show that Britt's formula works for a group of $k + 1$ people. Write her formula for the number of handshakes in a group of $k + 1$ people.

e. If an additional person enters a group of k people, how many new handshakes are necessary? Explain.

An expression for the total number of handshakes in a group of $k + 1$ people can be found by adding the expression for the number of handshakes in a group of k people (part c) to the number of additional handshakes when another person enters the group (part e): $\frac{k(k-1)}{2} + k$.

f. You can conclude that Britt's formula will work indefinitely if this expression is equivalent to the one you wrote in part d. Use algebraic procedures to transform the expression $\frac{k(k-1)}{2} + k$ until it matches the one you wrote in part d.

5. Although Britt's formula finds the number of handshakes in a group of people, it could also be used to find the number of potential two-party conflicts in a group.

a. Use the formula to compare the number of potential conflicts when the size of a group doubles. Does the number of potential conflicts also double? Explain.

b. Why do the results of Exercise 4 suggest that some of the costs associated with government, such as that of maintaining a police force, may outpace the growth of population?

In Exercises 1 through 4, you supplied many of the steps of the mathematical induction proof. In Exercise 6, which considers a formula for the relationship between the number of candidates on a ballot and the number of ways of casting an approval vote, you will again supply many of the steps of the induction process. However, before the proof can begin, you must do several preliminary steps leading to the conjecture of a formula. The preliminary steps are summarized here.

Preliminary Steps

Do the following before using mathematical induction to prove that a formula describes a relationship.

1. Organize a table of data for several small values. For example, how many ways of voting are there with 1, 2, 3, or 4 choices on the ballot?
2. Study the data and attempt to describe the data with a recurrence relation. For example, how many additional ways of voting are there when another choice is added to the ballot?
3. Conjecture a formula that predicts the outcome for a collection of k items. For example, what is a formula that predicts the number of ways of voting when there are k choices on the ballot?
4. Verify that your formula works for the values in your table.

6. In Exercise 12 of Lesson 1.4 (page 29), you considered the number of ways of voting under the approval system. The data you gathered in Lesson 1.4 are reproduced below. You will now use mathematical induction to verify a formula for the number of ways of voting under the approval system when there are n choices on the ballot.

Number of Choices on the Ballot	Number of Possible Ways of Voting
1	2
2	4
3	8
4	16

a. Collecting these data completes the first of the preliminary steps. The second step requires that you determine a recurrence relation that describes the relationship between the number of ways of voting when there are $k + 1$ choices on the ballot (V_{k+1}) and the number of ways of voting when there are k choices on the ballot (V_k). Do so, but be careful; if you do not establish the recurrence relation properly, the proof that comes later will fail. Here's a hint.

With Three Choices There are Eight Ways		The New Ways When a Choice Is Added
{ }		{D}
{A}		{A, D}
{B}	Append D	{B, D}
{C}	to each	{C, D}
{A, B}	→	{A, B, D}
{A, C}		{A, C, D}
{B, C}		{B, C, D}
{A, B, C}		{A, B, C, D}

b. You are ready for the third and fourth preliminary steps. Study the data carefully. Notice that the values in the second column are all powers of 2. What formula does this suggest for the number of ways of voting when there are n choices? Check the formula with each pair of values in the table.

Parts a and b complete the preliminary process. You are now ready to prove that the formula you conjectured in part b of Exercise 6 will work indefinitely.

The Proof

After you have conjectured a formula and shown that it works for a few cases, you can prove that it is correct by using mathematical induction:

1. State the meaning of your formula for a collection of size k. This is the assumption or hypothesis that you will use in your proof. (It is similar to the "given" in a geometric proof.)
2. State the meaning of your formula for a collection of size $k + 1$. You can do this by replacing k with $k + 1$ in the formula you wrote in step 1. This is the goal. (It is similar to the "prove" in a geometric proof.)
3. Use your recurrence relation to describe the effect of an additional object on the formula you stated in step 1.
4. Use algebraic procedures to transform the expression you wrote in step 3 until it matches the one you stated in step 2.

c. Assume that the formula you conjectured in part b works for a ballot with k choices: $V_k = 2^k$. You must show that the formula works for a ballot with $k + 1$ choices. Write the formula for a ballot with $k + 1$ choices to complete the first two steps of the proof process.

d. Write an expression for the total number of ways of voting on a ballot with $k + 1$ choices by applying the recurrence relation you gave in part a to the formula you stated in part b. Use algebraic procedures to show that the result is equivalent to your answer in part c. This completes steps 3 and 4 of the proof process, and your induction proof is finished.

In Exercises 7 to 12, collect and organize data into a table, examine the data, conjecture a formula, and use mathematical induction to prove that the formula is correct. If you need help with any of the steps, use the summaries of the preliminary steps and the proof in Exercise 6.

7. Dominoes come in sets of different sizes. A double-six set, for example, contains dominoes that pair every number of spots from 0 through 6 with itself and with every other number of spots. Find a formula for the number of dominoes in a double-k set.

8. Bowling pins are normally set in a triangular configuration. Find a formula for the number of pins in a triangular configuration of k rows.

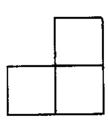
9. An ancient legend says that the inventor of the game of chess was offered a reward of his own choosing for the delight the game gave the king. The inventor asked for enough grains of wheat to be able to place one grain on the first square of the chessboard, two on the second, four on the third, and so forth, doubling the number of grains each time. Find a formula for the total number of grains on a chessboard after the kth square has been filled.

10. It takes four toothpicks to make a 1×1 square, and it takes 12 toothpicks to make a 2×2 square that is subdivided into 1×1 squares. Find a formula for the number of toothpicks needed to make a $k \times k$ square that is subdivided into 1×1 squares.

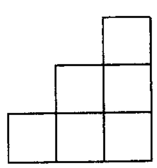
11. In Exercises 16 and 17 of Lesson 1.4 (page 31), you found a recurrence relation for the number of ways of selecting exactly two items when there are several choices on a ballot. Find a formula for the number of ways of selecting exactly two items when there are k choices on the ballot.

12. In the popular song "The Twelve Days of Christmas," one gift is given on the first day, one plus two on the second, and so on. Find a formula for the total number of gifts given on the kth day.

13. Find a formula for the number of building blocks in a set of $k \times k$ steps.



A 2×2 set of steps has 3 building blocks



A 3×3 set of steps has 6 building blocks.

14. Mathematicians often use mathematical induction to establish facts about numbers.

a. Use mathematical induction to prove that a formula for the kth odd integer is $2k - 1$.

b. Find and prove a formula for the sum of the first k odd integers. (Hint: The formula in part a may be of help.)

15. a. Find and prove a formula for the kth even integer.

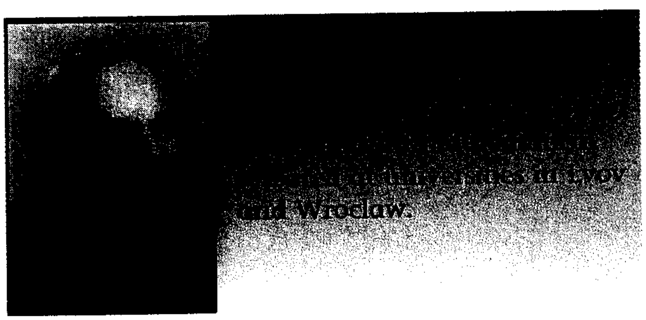
b. Find and prove a formula for the sum of the first k even integers.

Projects

16. Research and report on the use of mathematical induction in computer science.

Envy-Free Division

The first mathematician to take a serious interest in the problem of fairly dividing a cake may have been Hugo Steinhaus, a Polish mathematician who studied the problem during World War II. Steinhaus appears to have been motivated only by the challenge the problem posed. In the last half-century the world has served up an increasing number of problems that resemble cake division. Mathematicians and social scientists have risen to the occasion.



Steinhaus used the same definition of fairness that you used in this chapter: a cake is fairly divided among n people if each person feels that he or she has received at least $1/n$th of the cake. The cleverness of the people who have tackled fair division problems is demonstrated by the existence of an estate division algorithm that can result in everyone's getting more than a fair share by this definition.

However, the fairness definition is deficient in at least one important way. Fair division situations are sometimes charged with emotion. Not all participants behave as rationally as an impartial observer might expect. For example, according to this definition, a participant in a four-person estate division should be happy if the value placed on his or her received share is 0.3. But what if this individual feels someone else's share has a value of 0.35? Feeling that your share is fair is not the same as feeling it is larger than everyone else's.

For example, consider three individuals, A, B, and C, who are dividing a cake by the cut-and-choose method. Suppose that A and B are the cutters and that C places a value of 0.9 on A's piece and 0.1 on B's. Following the subsequent division of A's and B's pieces, C values A's three pieces at 0.3 each. C will now choose a piece valued at 0.3 from A, leaving A with two pieces C values at 0.3 each. Even if C gets all of B's pieces, C can do no better than 0.4 against A's 0.6.

The existence of situations in which envy arises suggests an alternative definition of fairness: Each person should feel that the received portion is at least as big as every other person's. The two-person cut-and-choose solution is free of envy, but the three-person cut-and-choose solution is not.

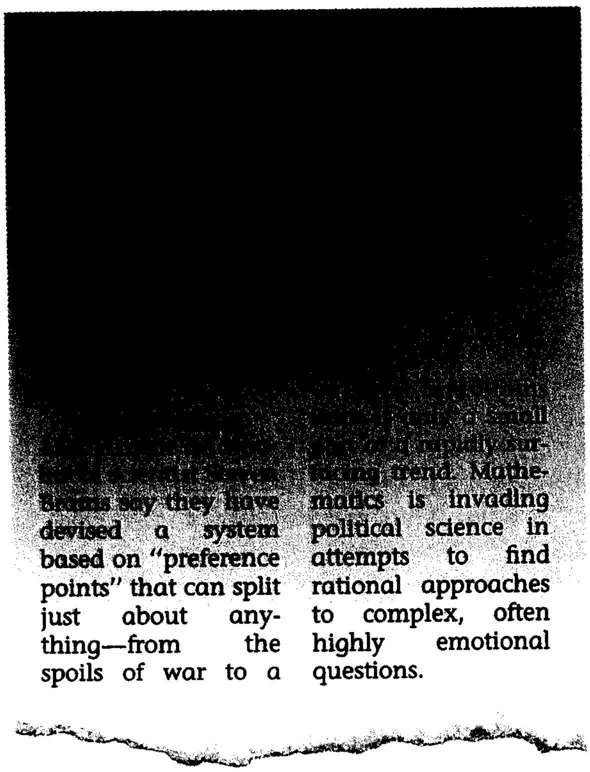
A three-person envy-free solution was devised independently by several mathematicians in the 1960s. It begins by having A cut the cake into three pieces. If B feels the division is not fair, B trims the best piece to make it the same value as the second-best. C gets first choice of one of the three pieces, B second choice, and A third. However, when B chooses, B must take the trimmed piece if C did not choose it. The trimmings can be disposed of by, say, feeding them to the dog. If the trimmings are substantial, they can be divided in a second round of cutting and choosing (although the roles of A, B, and C change).

Steven Brams is a political scientist at New York University.

Since C chooses first, C's piece will seem at least as good to C as either of the others. After C chooses, B is left at least one of the two pieces B considers better. Since B is forced to pick the trimmed piece if it is available, A is left one of the two pieces A cut on the first round.

Alan Taylor is a mathematician at Union College in Schenectady, New York.

None of the mathematicians who discovered the three-person solution succeeded in extending it to four or more. Then, in 1992, Steven Brams and Alan Taylor



Brams say they have devised a system based on "preference points" that can split just about anything—from the spoils of war to a

trend. Mathematics is invading political science in attempts to find rational approaches to complex, often highly emotional questions.

discovered an algorithm that produces an envy-free solution for any number of people.

Although the method of Brams and Taylor applies to a divisible object such as a cake, the pair quickly adapted it to situations in which only a portion of the goods is divisible (i.e., the cash in an estate). In a recent book on fair division, Brams and Taylor apply the method to a wide range of real-world fair division problems, including situations, such as the division of chores, in which the "goods" are really "bads."

Chapter 2 Review

1. Write a summary of what you think are the important points of this chapter.

2. Joan, Henry, and Sam are heirs to an estate that includes a vacant lot, a boat, a computer, a stereo, and $10,000 in cash. Each heir has submitted bids as summarized in the following table.

	Joan	Henry	Sam
Vacant lot	$8,000	$7,500	$6,200
Boat	$6,500	$5,700	$6,700
Computer	$1,340	$1,500	$1,400
Stereo	$800	$1,100	$1,000

For each heir, find the fair share, the items received, the amount of cash, and the final settlement. Summarize your results in a matrix.

3. Anne, Beth, and Jay are heirs to an estate that includes a computer, a used car, and a stereo. Each heir has submitted bids for the items in the estate as summarized in the following table.

	Anne	Beth	Jay
Computer	$1,800	$1,500	$1,650
Car	$2,600	$2,400	$2,000
Stereo	$1,000	$800	$1,200

For each heir, find the fair share, the items received, the amount of cash, and the final settlement. Summarize your results in a matrix.

4. States A, B, and C have populations of 647, 247, and 106, respectively. There are 100 seats to be apportioned among them.

a. What is the ideal ratio?
b. Find the quota for each state.
c. Apportion the 100 seats among the three states by the Hamilton method.
d. What is the initial Jefferson apportionment?
e. Find the Jefferson adjusted ratio for each state.
f. Apportion the 100 seats by the Jefferson method.
g. What is the initial Webster apportionment?
h. Find the Webster adjusted ratio for each state.
i. Apportion the 100 seats by the Webster method.
j. What is the initial Hill apportionment?
k. Find the Hill adjusted ratio for each state.
l. Apportion the 100 seats by the Hill method.
m. Suppose the populations of the states change to 650, 255, and 105, respectively. Reapportion the 100 seats by the Hamilton method.
n. Explain why the results in part m constitute a paradox.

5. Discuss the theorem proved by Michel Balinski and H. Peyton Young. That is, what did they prove?

6. Arnold, Betty, and Charlie are dividing a cake in the following way.

Arnold divides the cake into what he considers six equal pieces.

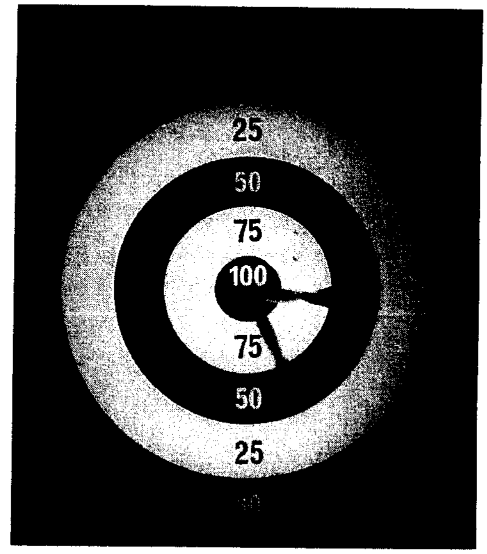
The pieces are then chosen in this order: Betty, Charlie, Betty, Charlie, Arnold, Arnold.

Who is guaranteed a fair share by his or her own assessment?

7. Four people have divided a cake into four pieces that each considers fair, and then a fifth person arrives. Describe a method of dividing the four existing pieces so that each of the five people receives a fair share.

8. For the following situation, collect and organize data into a table, examine the data and conjecture a formula, and use mathematical induction to prove that your formula is correct.

   In a set of concentric circles, a ring is any region that lies between any two of the circles. Find a formula for the number of rings in a set of k concentric circles.



9. On the basis of the enrollment in each of a high school's courses, the administration must decide the number of sections that are offered. A number of factors affect the decision. For example, financial considerations require about 25 students in each section and a maximum of 300 sections for all courses. Recommend a procedure the school might use to divide 300 sections fairly among all courses on the basis of the enrollment in those courses.

10. Discuss how fair division methods you studied in this chapter might change to accommodate a situation in which the objects being divided were undesirable (i.e., the division of household chores among children).

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