CHAPTER

Matrix Operations and Applications

3

It has been said that sports fans are the nation's foremost consumers of statistics and that baseball fans are the most prominent among them. Whether it's in the information conveyed by a scoreboard, as on this one that commemorates the 1941 National League pennant race, or in the current information on league leaders, baseball records are full of numbers.

How can large collections of data be organized and managed in an efficient way? What calculations provide meaningful information to people who use the data? How can computers and calculators assist them? Baseball statisticians, business executives, and wildlife biologists are among the diverse groups of people who turn to the mathematics of matrices for answers to these questions.

Lesson 3.1

Addition and Subtraction of Matrices

In Lesson 2.2, you were introduced to matrices as a natural way to organize, manipulate, and display information. As you have seen, matrices provide a very handy device for representing sets of discrete data that can be described with two characteristics, one characteristic being represented by the rows of the matrix and the other by the columns. In this lesson you will be introduced to some of the terminology and notation used in working with matrices. Matrix addition and subtraction will also be developed.

Matrix Terminology

As a first example, suppose that you and a few of your friends are planning a pizza and video party and some decisions have to be made about ordering food. You decide to call the pizza houses that deliver in your neighborhood to ask about prices for large single-topping pizzas, liter containers of cold drinks, and family-sized salads with house dressing. You could record the information you get in a table such as the following.

	Gina's	Vin's	Toni's	Sal's
Pizza	$12.16	$10.10	$10.86	$10.65
Drinks	$1.15	$1.09	$0.89	$1.05
Salad	$4.05	$3.69	$3.89	$3.85

Or you might choose to write your data in **matrix form,** which simply means writing the numbers in a rectangular array and enclosing them in brackets or parentheses.

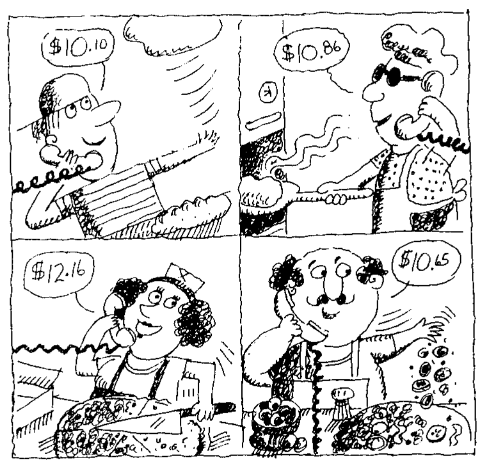
$$\begin{array}{r} \\ \text{Pizza} \\ \text{Drinks} \\ \text{Salad} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Gina's} & \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ \begin{bmatrix} \$12.16 & \$10.10 & \$10.86 & \$10.65 \\ \$1.15 & \$1.09 & \$0.89 & \$1.05 \\ \$4.05 & \$3.69 & \$3.89 & \$3.85 \end{bmatrix} \end{array}$$

For the sake of simplicity, you could omit the row and column labels and dollar signs in your matrix and write only the values. If you delete the labels, however, you will have to remember that the rows represent the prices for pizzas, drinks, and salads, while the columns represent the various pizza houses.

$$\begin{bmatrix} 12.16 & 10.10 & 10.86 & 10.65 \\ 1.15 & 1.09 & 0.89 & 1.05 \\ 4.05 & 3.69 & 3.89 & 3.85 \end{bmatrix}$$

Each of the individual entries in the matrix is called an **element** or a **component** of the matrix. A matrix, such as this one, with three rows and four columns is described as a matrix with **order** or **dimension** 3 by 4 (written as 3×4). It is also customary to refer to this matrix as simply a 3×4 matrix.

In general, if a matrix has *m* rows and *n* columns, in which *m* and *n* represent counting numbers, it is called an *m* by *n matrix.*



After looking over your data, you might decide to drop Gina's options from the possible choices since they are more expensive item by item than any of the others. If you do this, you will be left with a 3×3 **square matrix.** Notice that in a square matrix the number of rows equals the number of columns or $m = n$.

$$\begin{array}{l} \\ \text{Pizza} \\ \text{Drinks} \\ \text{Salad} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ \begin{bmatrix} \$10.10 & \$10.86 & \$10.65 \\ \$1.09 & \$0.89 & \$1.05 \\ \$3.69 & \$3.89 & \$3.85 \end{bmatrix} \end{array}$$

Matrices do not always have multiple rows and columns. For example, if your group thinks about which pizza most people prefer, regardless of the price, you might decide that it would be Sal's. If you list only the prices for Sal's offerings, the result will be a **column matrix** of order 3 × 1. Matrices that have only one column are sometimes referred to as **column vectors.**

$$\begin{array}{l} \\ \text{Pizza} \\ \text{Drinks} \\ \text{Salad} \end{array} \begin{array}{c} \text{Sal's} \\ \begin{bmatrix} 10.65 \\ 1.05 \\ 3.85 \end{bmatrix} \end{array}$$

If you choose to look at the pizza prices alone, they can be represented with a 1 × 3 **row matrix,** or **row vector.**

$$\begin{array}{l} \\ \text{Pizza} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ \begin{bmatrix} \$10.10 & \$10.86 & \$10.65 \end{bmatrix} \end{array}$$

Notice that when you give the order of a matrix, you write the number of rows followed by the number of columns. The simplest order would be that of a **1 × 1 matrix** such as [10.65] containing a single element.

Exercises

1. Write a definition for a matrix in your own words. Justify the validity of your definition through discussion with other students in your class.

2. Bring to class at least two matrices from newspapers or magazines (such as the one in the newsclip on page 112).
   a. What are the dimensions of each of your matrices?
   b. What is represented by the rows and columns of your matrices?
   c. Be prepared to share your matrices with other class members.

3. A trendy garment company receives orders from three boutiques. The first boutique orders 25 jackets, 75 shirts, and 75 pairs of pants. The second boutique orders 30 jackets, 50 shirts, and 50 pairs of pants. The third boutique orders 20 jackets, 40 shirts, and 35 pairs of pants. Display this information in a matrix whose rows represent the boutiques and whose columns represent the type of garment ordered. Label the rows and columns of your matrix accordingly.

4. Although matrices contain many data values, they can also be thought of as single entities. This feature allows us to refer to a matrix with a single capital letter:

$$A = \begin{array}{r} \\ \text{Pizza} \\ \text{Drinks} \\ \text{Salad} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ \begin{bmatrix} \$10.10 & \$10.86 & \$10.65 \\ \$1.09 & \$0.89 & \$1.05 \\ \$3.69 & \$3.89 & \$3.85 \end{bmatrix} \end{array}$$

or

$$S = \text{Pizza} \begin{array}{c} \begin{array}{ccc} \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ [\$10.10 \quad \$10.86 \quad \$10.65] \end{array}$$

Individual entries in a matrix are identified by row number and column number, in that order. For example, the value 10.65 is the entry in row 1 and column 3 of matrix A and is referenced as A_{13}. Entry A_{13} represents or *is interpreted as* the cost of a pizza at Sal's. Notice that A_{31} is not the same as A_{13}. Entry A_{31} has the value \$3.69 and represents the cost of a salad at Vin's. In the row matrix S, the entries are referenced as S_1, S_2, and S_3.

a. What is the value of A_{21}? Of A_{12}? Of A_{32}?
b. Write an interpretation of each of the entries in part a.
c. Write an interpretation of S_3.

5. For breakfast Patty had cereal, a medium banana, a cup of 2% milk, and a slice of buttered toast. She recorded the following information in her food journal. Cereal: 165 calories, 3 g fat, 33 g carbohydrate, and no cholesterol. Banana: 120 calories, no fat, 26 g carbohydrate, and no cholesterol. Milk: 120 calories, 5 g fat, 11 g carbohydrate, and 15 mg cholesterol. Buttered toast: 125 calories, 6 g fat, 14 g carbohydrate, and 18 mg cholesterol.

a. Write this information in a matrix N whose rows represent the foods. Label the rows and columns of your matrix.
b. State the values of N_{23}, N_{32}, and N_{42}.
c. Write an interpretation of N_{23}, N_{32}, and N_{42}.

6. Suppose that as you continue to plan your pizza party, you discover that the local supermarket has a sale on 2-liter bottles of soft drinks and you decide not to order drinks from a pizza house after all. Write and label a 2×3 matrix that represents the prices for just pizza and salad at Vin's, Toni's, and Sal's.

7. Suppose further that when you were calling the pizza houses about prices, you also collected the following information about the cost of additional toppings and salad dressings.

	Vin's	Toni's	Sal's
Additional toppings	$1.15	$1.10	$1.25
Additional dressings	$0.00	$0.45	$0.50

Represent the information from this table in another 2×3 matrix whose rows represent the additional toppings and dressings and whose columns represent the three pizza houses. Label the rows and columns of your matrix.

8. Your next step is to compute what it would cost to order pizzas with two toppings and to allow a choice of two salad dressings. This can be done by simply adding corresponding components of your two price matrices. If you let A represent the basic price matrix and B represent the matrix of additional costs, then you can add A and B to get a third matrix C, which will represent the total prices for pizza and salads at each pizza house.

$$A = \begin{bmatrix} 10.10 & 10.86 & 10.65 \\ 3.69 & 3.89 & 3.85 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1.15 & 1.10 & 1.25 \\ 0.00 & 0.45 & 0.50 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 10.10 + 1.15 & 10.86 + 1.10 & \_\_\_\_\_ \\ \_\_\_\_\_ & \_\_\_\_\_ & \_\_\_\_\_ \end{bmatrix}$$
$$= \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix} = C.$$

Complete the addition and label the rows and columns of matrix C.

9. In Exercise 8, the entries of matrix C represent the sum of the corresponding entries in matrices A and B. For example, C_{13}, which represents the cost of a pizza with an extra topping at Sal's, equals the sum of A_{13} and B_{13}.

a. What is the value of A_{21}? Of B_{21}? Of C_{21}?

b. Write an interpretation of A_{21}, B_{21}, and C_{21}.

In general, if A and B are m by n matrices, then $C = A + B$ is a matrix whose entries represent the sum of the corresponding entries in matrices A and B. This matrix sum is represented with symbols as

$$C_{ij} = A_{ij} + B_{ij} \text{ where } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

It is clear from this discussion that matrices of unlike dimensions can not be added. It also does not make sense to add matrices whose row and column labels represent unlike quantities.

10. Suppose that a physician associates with each patient a row matrix whose components represent that person's age, weight, and height. Would it be appropriate to add together the matrices associated with two different patients? Explain your answer.

11. Suppose that the manager of a convenience store associates with each customer a column vector whose components represent the customer's purchases. Would it make sense to add together the vectors representing the purchases of two or more customers? Explain your answer.

12. Through July 20, 1997, the three baseball players with the highest batting averages in the National League had the following batting statistics.

	AB	R	H	HR	RBI	Avg
L. Walker (Colorado)	343	88	138	27	79	.402
Gwynn (San Diego)	372	64	147	15	84	.395
Piazza (Los Angeles)	332	56	118	19	62	.355

The following statistics for the same three players were published on September 30, 1997.

	AB	R	H	HR	RBI	Avg
L. Walker (Colorado)	568	143	208	49	130	.366
Gwynn (San Diego)	592	97	220	17	119	.372
Piazza (Los Angeles)	556	104	201	40	124	.362

At the end of the 1997 season only Ty Cobb had won more batting titles (12) than the Padres' Tony Gwynn (8).

Find and label a matrix that displays the changes in these statistics over the course of the season. Notice that the batting averages for two of the three player's decreased from July through September. How will you show this in your matrix?

13. Write a definition for **matrix subtraction** based on your calculations in Exercise 12. Justify the validity of your definition through discussion with your classmates.

14. The matrices that follow give the average times in minutes for physical fitness test endurance runs for U.S. youth in 1980 and 1989 (U.S. Department of Education, 1991). The runs were 3/4 mile for ages 10 and 11 and 1 mile for ages 12 through 17.

	1980 Boys	1980 Girls	1989 Boys	1989 Girls
10- and 11-year-olds	6.5	7.4	7.3	8.0
12- and 13-year-olds	8.4	9.8	9.1	10.5
14- and 17-year-olds	7.5	9.6	8.6	10.7

a. Find and label the matrix that represents the change in times in minutes for each age group from 1980 to 1989.
b. In which age group and sex was there the greatest increase in average time for the endurance runs? The smallest increase?
c. How would you explain this increase in times across age groups?

15. In statistics, a correlation matrix is a matrix whose entries represent the degree of relationship between variables. The values in a correlation matrix range from -1 to 1, where 0 indicates that there is no relationship, a negative value indicates that as one variable increases the other decreases, and a positive value indicates that as one variable increases the other one also increases. In a study of the relationship between ACT test scores, high school class rank, and college grade point average, the following correlation matrix was generated. Notice that the row labels and the column labels are the same in a correlation matrix.

	ACT Comp.	ACT Eng.	ACT Math	ACT Soc. St.	ACT Sci.	H.S. Rank	Coll. GPA
ACT Composite	1.00	0.80	0.79	0.81	0.82	0.59	0.51
ACT English	0.80	1.00	0.54	0.58	0.55	0.53	0.48
ACT Math	0.79	0.54	1.00	0.42	0.52	0.57	0.42
ACT Social studies	0.81	0.58	0.42	1.00	0.61	0.39	0.39
ACT Science	0.82	0.55	0.52	0.61	1.00	0.44	0.36
High school rank	0.59	0.53	0.57	0.39	0.44	1.00	0.45
College GPA	0.51	0.48	0.42	0.39	0.36	0.45	1.00

Source: Aksamit, Mitchell, and Pozehl, 1986.

The ones in this matrix are located along what is called the **main diagonal**, in which the row and column numbers are the same. If we call this matrix R, then these diagonal elements are referenced as R_{ii} ($i = 1, 2, \ldots, 7$). We say that this matrix is **symmetric**, since $R_{ij} = R_{ji}$ ($i, j = 1, 2, \ldots, 7$).

In a symmetric matrix you need only to know the values along the main diagonal and either the triangle above the main diagonal (the upper triangle) or below it (the lower triangle). Because of this feature, correlation matrices are often written with blanks in either the upper or lower triangle.

a. Why do you think the values along the main diagonal of a correlation matrix are all 1s?
b. Could a matrix that is not square be symmetric? Why?
c. Why are the values in a correlation matrix symmetrical about the main diagonal?
d. Which variable had the highest correlation with college GPA?
e. Which subject area test had the highest correlation with high school rank?

16. In your study of algebra, you learned that the commutative and associative properties hold for addition over the set of real numbers (That is, for all real numbers a, b, and c, $a + b = b + a$ and $a + (b + c) = (a + b) + c$.

Campus Administrators Announce New Parking Plans

SCARLET,
March 11, 1999

Campus planners are working overtime to finalize plans for three new parking garages and a beefed up shuttle system to replace parking stalls that will be lost in the next four to five years due to construction. Administrators acknowledge that the new parking and transit plan has a downside: parking fees will increase significantly. It will cost about $50 million to build the new garages and support the shuttle system. These costs will be borne by the users. Typical faculty/staff permits will escalate in $5 or $6 monthly increments annually through 2004. Student fees will also increase, although in $4 or $5 increments. Administrators admit that this will be painful and that the rate of escalation will leave some folks breathless.

Permit Fees

Year	Structure Completed	F/S/S Monthly Perimeter Permits	Monthly Student Permits	Monthly F/S Permits	Monthly Student Reserved	Monthly F/S Reserved
1998		$4	$9	$11	$26	$31
1999		$5	$13	$17	$35	$40
2000		$6	$17	$23	$45	$50
2001	South	$7	$21	$29	$50	$60
2002	East	$8	$25	$34	$55	$70
2003		$9	$29	$39	$60	$80
2004	North-1	$10	$32	$44	$65	$90

a. Do you think that the commutative and associative properties hold for addition of matrices? Why?

b. Use the following matrices to test your conjecture in part a.

$$A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}.$$

17. Do you think that the commutative and associative properties hold for subtraction of matrices? Why? Test your conjecture using matrices A, B, and C in Exercise 16b.

18. A matrix all of whose entries are the number zero is called a **zero matrix** and is denoted using a capital letter O alone or with subscripts $O_{m \times n}$.

a. Show that $A + O = O + A = A$ and that $A - A = O$.

b. Show that $A + (-A) = (-A) + A = O$, where the matrix $-A$, called the negative of A, is obtained by negating each entry in A.

Lesson 3.2

Multiplication of Matrices, Part 1

In the previous lesson, matrix addition and subtraction were defined by looking at some matrix models of real-world situations. In this lesson, matrix multiplication is approached in much the same manner.

Multiplication by a Scalar

Examine again the data in the pizza problem posed in Lesson 3.1. Suppose that a decision that the group must make is how many of each type of pizza and salad it can afford to order. You can start this decision-making process by computing the cost of ordering four of each of the pizzas and salads represented by matrix C in Exercise 8 of Lesson 3.1. To do this, multiply each element in matrix C by 4 to get a new matrix T, equal to $4C$. An operation of this type is called **multiplication of a matrix by a scalar**. Multiplication of a matrix by a scalar is analogous to multiplication of integers in that $4C$ could also be interpreted as repeated addition or $4C = C + C + C + C$.

$$4C = 4 \times \begin{bmatrix} 11.25 & 11.96 & 11.90 \\ 3.69 & 4.34 & 4.35 \end{bmatrix} = \begin{bmatrix} 4(11.25) & 4(11.96) & 4(11.90) \\ 4(3.69) & 4(4.34) & 4(4.35) \end{bmatrix}$$

$$= \begin{bmatrix} 45.00 & 47.84 & 47.60 \\ 14.76 & 17.36 & 17.40 \end{bmatrix} = T.$$

Labeling the rows and columns of the matrix, you have

$$T = \begin{array}{c} \\ \text{Pizza} \\ \text{Salad} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ \begin{bmatrix} \$45.00 & \$47.84 & \$47.60 \\ \$14.76 & \$17.36 & \$17.40 \end{bmatrix} \end{array}$$

Multiplication by a Row Matrix

Multiplication of a matrix by a scalar is only one way in which multiplication can be applied in matrix situations. **Multiplication of a column matrix by a row matrix** is illustrated in the following problem. Suppose that Jon, a student at Washington High, runs out to the nearby Super X to buy some junk food to stock up his locker for between-class snacks. He chooses four small bags of chips, five candy bars, a box of cheese crax, three packs of sour drops, and two bags of cookies. Jon's purchases can be represented by a row matrix Q.

$$Q = \begin{array}{c} \begin{array}{ccccc} \text{Chips} & \text{Candy} & \text{Crax} & \text{Drops} & \text{Cookies} \end{array} \\ \begin{bmatrix} 4 & 5 & 1 & 3 & 2 \end{bmatrix} \end{array}.$$

Suppose further that chips cost 30 cents a bag, candy bars cost 35 cents each, crax cost 50 cents a box, sour drops cost 20 cents a pack, and cookies sell for 75 cents a bag. These prices can be represented in column matrix P.

$$P = \begin{array}{c} \text{Chips} \\ \text{Candy} \\ \text{Crax} \\ \text{Drops} \\ \text{Cookies} \end{array} \begin{array}{c} \text{Cents} \\ \begin{bmatrix} 30 \\ 35 \\ 50 \\ 20 \\ 75 \end{bmatrix} \end{array}.$$

The obvious question to ask now is, "How much did Jon pay for all these goodies?" To find the answer, multiply the price vector P by the quantity vector Q.

$$Q \times P = [4 \quad 5 \quad 1 \quad 3 \quad 2]\begin{bmatrix}30\\35\\50\\20\\75\end{bmatrix}$$

$$= [4(30) + 5(35) + 1(50) + 3(20) + 2(75)]$$
$$= [120 + 175 + 50 + 60 + 150]$$
$$= [555] \text{ cents} = [\$5.55].$$

This matrix computation is, of course, exactly what the clerk at Super X would do in figuring Jon's bill. The price of each item is multiplied by the number purchased and the products are summed. In order to do this computation, it is obvious that the number of items and the number of prices must be the same. Items and prices must also correspond.

> **In general, if Q is a row matrix and P is a column matrix, each having the same number of components, then the product QP is defined. The matrix product Q times P is found by multiplying the corresponding components and summing the results.**

As a second example, suppose another student, Trilby, goes along with Jon to the Super X. Her purchases are a bag of chips, two candy bars, two packs of gum that cost 25 cents each, and a medium drink for 75 cents.

1. Write and label a row matrix Q that represents the quantity of each item that Trilby purchased and a column matrix P that represents the price for each item.
2. Perform the multiplication Q times P to find the total cost of Trilby's purchases.
3. Compare your work with that of other students in your class. Was your final matrix [$2.25]?

As you can see from these examples, when a column matrix is multiplied by a row matrix the result is a single value. In other words, when a $(k \times 1)$ column matrix P is multiplied by a $(1 \times k)$ row matrix

Q, the result is a (1×1) single-value matrix C. Schematically this product looks like

$$\underset{(1 \times k)}{Q} \times \underset{(k \times 1)}{P} = \underset{(1 \times 1)}{C.}$$

Same

Dimensions of the product

In the previous examples you saw how multiplication of a matrix by a scalar and multiplication of a column matrix by a row matrix are defined. The next step is to define **multiplication of a matrix with more than one column by a row matrix.** This type of matrix multiplication is illustrated by examining some additional options in the pizza problem.

Suppose your group decides to order five pizzas and three salads and you want to know the total cost at each of the pizza houses for this combination. If you do the calculations involved here without using matrices, you proceed by multiplying the pizza price by 5 and adding the result to 3 times the salad price for each pizza house as follows.

Cost at Vin's: 5(\$11.25) + 3(\$3.69) = \$56.25 + \$11.07 = \$67.32.

Cost at Toni's: 5(\$11.96) + 3(\$4.34) = \$59.80 + \$13.02 = \$72.82.

Cost at Sal's: 5(\$11.90) + 3(\$4.35) = \$59.50 + \$13.05 = \$72.55.

To use matrix multiplication to solve this problem, the number of pizzas and salads you plan to order are modeled by the 1×2 row matrix A.

$$A = \begin{array}{cc} \text{Pizzas} & \text{Salads} \\ [\ 5 & 3\ ] \end{array}.$$

The prices for pizzas and salads at each of the three pizza houses are modeled by the 2×3 matrix C.

$$C = \begin{array}{c} \\ \text{Pizzas} \\ \text{Salads} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ \begin{bmatrix} 11.25 & 11.96 & 11.90 \\ 3.69 & 4.34 & 4.35 \end{bmatrix} \end{array}.$$

Now when matrix C is multiplied by the row matrix A, the expected result is another matrix whose components will give the total cost for five pizzas and three salads at each pizza house. To accomplish this, it makes sense to multiply each of the columns in matrix C by the row matrix A:

$$A \times C = [5 \quad 3] \times \begin{bmatrix} 11.25 & 11.96 & 11.90 \\ 3.69 & 4.34 & 4.35 \end{bmatrix}$$

$$= [5(11.25) + 3(3.69) \quad 5(11.96) + 3(4.34) \quad 5(11.90) + 3(4.35)]$$
$$= [56.25 + 11.07 \quad 59.80 + 13.02 \quad 59.50 + 13.05]$$
$$= [67.32 \quad 72.82 \quad 72.55].$$

The last step is to label the entries in the final product:

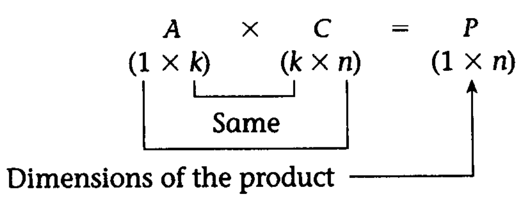
Vin's	Toni's	Sal's
[\$67.32	\$72.82	\$72.55].

Notice that in carrying out this matrix computation, the process was exactly the same as that of figuring the costs without the use of matrices. In the matrix multiplication, each of the components in the columns of matrix C was multiplied by the corresponding component the row matrix A. These products were then summed to give the components of the final product matrix.

You can see from this model that matrix multiplication of this sort can only be defined when the number of entries in the row matrix equals the number of rows in the multidimensional matrix. In addition, the product will be a row matrix with the same number of entries as there are columns in the second matrix.

> In general, the product of a $(1 \times k)$ row matrix A and a $(k \times n)$ matrix C is a $(1 \times n)$ row matrix P.

This result can be represented schematically as follows.



Another example: Suppose your group wants to look at a couple of other combinations of pizzas and salads before it makes its final decision about how many to order. Use matrix multiplication to calculate the totals for (1) four pizzas and three salads and (2) four pizzas and four salads at each of the three pizza houses. Be sure to label your matrices. Check the steps in your work and discuss the interpretations of each of your answers with other students in your class. Did you get (1) [$56.07 $60.86 $60.65] and (2) [$59.76 $65.20 $65.00]?

Exercises

1. Refer to matrix T in the first example of this lesson.
   a. What does matrix T represent?
   b. What is the cost of four pizzas at Sal's?
   c. Interpret T_{12} and T_{21}.

2. Nancy has a small shop in the Oldmarket where she makes and sells four different kinds of jewelry: earrings (e), pins (p), necklaces (n), and bracelets (b). She fashions each item out of either cultured pearls or jade beads. The following matrix represents Nancy's sales for May.

$$M = \begin{array}{l} \\ \text{Pearl} \\ \text{Jade} \end{array} \begin{array}{c} \begin{array}{cccc} \text{e} & \text{p} & \text{n} & \text{b} \end{array} \\ \begin{bmatrix} 8 & 4 & 6 & 5 \\ 20 & 10 & 12 & 9 \end{bmatrix} \end{array}.$$

Nancy hopes to sell twice as many of each piece in June.
   a. Calculate a matrix J, where $J = 2M$ to represent the number of each item Nancy will sell in June if she reaches her goal.
   b. Label the rows and columns of matrix J.
   c. How many jade necklaces does Nancy expect to sell in June?
   d. Interpret J_{21} and J_{12}.

3. Matt reads on the side of his cereal box that each ounce of cereal contains the following percentages of the minimum daily requirements of:

Vitamin A	25%
Vitamin C	25%
Vitamin D	10%

If Matt eats 3 ounces of cereal for breakfast, what percentages of each vitamin will he get? Show the matrices and matrix operation involved in your calculation. Label your matrices.

4. The regents at a state university recently announced a 7% raise of tuition rates per semester hour. The current rates per semester hour are shown in the following table.

	Undergraduate	Graduate
Resident	\$75.00	\$99.25
Nonresident	\$204.00	\$245.25

a. Write and label a matrix that represents this information.
b. Find a new matrix that represents the tuition rates per semester hour after the 7% raise goes into effect. Label your matrix.
c. Find a matrix that represents the dollar increase for each of the categories. Label your matrix.
d. Which matrix multiplication operation did you use in part b?

5. As you saw in the second example in this lesson (page 115), if Q is a (1×5) row matrix and P is a (5×1) column matrix, then the product Q times P is a single-value matrix with dimension (1×1). Suppose you multiply the column matrix P times the row matrix Q.
a. What will be the dimension of the product P times Q? Justify your answer using a schematic diagram similar to the ones following the examples in this lesson.
b. Multiply matrix P times matrix Q. (Refer to the second example in this lesson for the values in these matrices.)
c. Does the product P times Q have a meaningful interpretation in this situation? Explain your answer.

6. Teresita's credit union has investments in three states—Massachusetts, Nebraska, and California. The deposits in each state are divided between consumer loans and bonds.

The amount of money (in thousands of dollars) invested in each category is displayed in the following table.

	Mass.	Neb.	Cal.
Loans	230	440	680
Bonds	780	860	940

The current yields on these investments are 6.5% for consumer loans and 7.2% for bonds. Use matrix multiplication to find the total earnings for each state. Label your matrices.

7. Nancy is doing some remodeling of her home. She makes a trip to the lumber company to pick up ten 2 × 6s, four 4 × 6s, and two 5 × 5s. In 8-foot lengths, 2 × 6s cost $3.00, 4 × 6s cost $8.50, and 5 × 5s cost $9.50.

a. Write and label a row matrix and a column matrix to represent the information in this problem.
b. Will everyone necessarily write the same row and column matrices? Explain your answer.
c. Perform a matrix multiplication to find the total cost of Nancy's purchases.

8. Chiu has $10,000 in a 12-month CD at 7.3% (annual yield), $17,000 in a credit union at 6.5%, and $12,000 in bonds at 7.5%. Use vector multiplication to find Chiu's earnings for a year. Label your vectors.

9. The **transpose** (A^T) of a matrix A is the matrix obtained by interchanging the rows and columns of matrix A.

a. Describe the transpose of a row matrix and of a column matrix.
b. Write and label the transpose (M^T) of matrix M in Exercise 2.
c. What might be a reason for wanting to know the transpose of a matrix?

10. Refer to Exercise 2. Suppose it takes Nancy 2 hours to make a pair of earrings, 1 hour to make a pin, 2.5 hours to make a necklace, and 1.5 hours to make a bracelet.

a. Write and label a row matrix that represents this information.
b. Use matrix multiplication to find a matrix that represents the total hours Nancy spends making each type of jewelry (cultured pearls or jade) for the month of May. (Hint: Use the transpose of matrix M that you found in the previous exercise.)
c. Label your product matrix.
d. Interpret each of the entries in the product matrix.

11. Suppose Nancy has expanded her jewelry business and now has shops in the Westmarket and Eastmarket plazas as well as in the Oldmarket. Her sales of cultured pearl sets for July are shown in the following table.

	Old	West	East
Earrings	10	8	12
Pins	6	5	4
Necklaces	3	2	2
Bracelets	4	3	2

Earrings sell for \$40 a pair, pins for \$35 each, necklaces for \$80, and bracelets for \$45. Use matrix multiplication to find Nancy's total sales at each location. Label your matrices.

12. During the first week of a recent fund-raiser for the math club at Washington High, Anne sold the following number of candy canes.

	Mon	Tues	Wed	Thurs	Fri
Canes	10	15	20	30	50

a. Write this information in a column matrix C. Label your matrix.
b. Find a row matrix N such that the product N times C gives the total number of candy canes that Anne sold for the week.
c. Find a row matrix A such that the product A times C gives the average number of candy canes that Anne sold each day. (Hint: What fraction would you multiply the total number of canes by to find the average?)

Lesson 3.3

Multiplication of Matrices, Part 2

In this lesson, you will continue to explore matrix multiplication by looking at the products of multidimensional matrices. There was an example of this type of matrix multiplication in the exercises of Lesson 2.2. Now, if you look at the three multiplications done in Lesson 3.2 to compare the cost of different combinations of pizza and salad, it seems to make sense to combine all three options into a single matrix *B* and to perform a single matrix multiplication.

$$B = \begin{array}{r} \\ \text{Option 1} \\ \text{Option 2} \\ \text{Option 3} \end{array} \begin{array}{c} \begin{array}{cc} \text{No. pizzas} & \text{No. salads} \end{array} \\ \begin{bmatrix} 4 & 3 \\ 4 & 4 \\ 5 & 3 \end{bmatrix} \end{array}$$

If you multiply matrix *B* times matrix *C* (see Lesson 3.2), the product (call it *D*) is a 3×3 matrix whose rows represent the three options and whose columns represent the three pizza houses. The components of this matrix give the total cost for each of the three options at each of the three pizza houses.

Notice as you follow the steps of this matrix multiplication that the computations are exactly the same as shown in the three separate calculations in the previous lesson. You expect, then, that row 1 of the product represents the cost of four pizzas and three salads, that row 2 of the product represents the cost of four pizzas and four salads, and that row 3 of the product represents the cost of five pizzas and three salads at each of the pizza houses.

Then, matrix B times matrix C =

$$\begin{array}{r} \\ \text{Option 1} \\ \text{Option 2} \\ \text{Option 3} \end{array}\begin{array}{c} \begin{array}{cc} \text{Pizzas} & \text{Salads} \end{array} \\ \begin{bmatrix} 4 & 3 \\ 4 & 4 \\ 5 & 3 \end{bmatrix} \end{array} \times \begin{array}{r} \\ \text{Pizzas} \\ \text{Salads} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ \begin{bmatrix} 11.25 & 11.96 & 11.90 \\ 3.69 & 4.34 & 4.35 \end{bmatrix} \end{array}$$

or

$$\begin{bmatrix} 4 & 3 \\ 4 & 4 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 11.25 & 11.96 & 11.90 \\ 3.69 & 4.34 & 4.35 \end{bmatrix}$$

$$= \begin{bmatrix} 4(11.25) + 3(3.69) & 4(11.96) + 3(4.34) & 4(11.90) + 3(4.35) \\ 4(11.25) + 4(3.69) & 4(11.96) + 4(4.34) & 4(11.90) + 4(4.35) \\ 5(11.25) + 3(3.69) & 5(11.96) + 3(4.34) & 5(11.90) + 3(4.35) \end{bmatrix}$$

$$= \begin{bmatrix} 45.00 + 11.07 & 47.84 + 13.02 & 47.60 + 13.05 \\ 45.00 + 14.76 & 47.84 + 17.36 & 47.60 + 17.40 \\ 56.25 + 11.07 & 59.80 + 13.02 & 59.50 + 13.05 \end{bmatrix}$$

$$= \begin{bmatrix} 56.07 & 60.86 & 60.65 \\ 59.76 & 65.20 & 65.00 \\ 67.32 & 72.82 & 72.55 \end{bmatrix} = D.$$

If you label the product matrix for clarity's sake, the result is

$$D = \begin{array}{r} \\ \text{Option 1} \\ \text{Option 2} \\ \text{Option 3} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Vin's} & \text{Toni's} & \text{Sal's} \end{array} \\ \begin{bmatrix} 56.07 & 60.86 & 60.65 \\ 59.76 & 65.20 & 65.00 \\ 67.32 & 72.82 & 72.55 \end{bmatrix} \end{array}.$$

In this matrix, D_{11} represents the cost of four pizzas and three salads at Vin's. How would you interpret D_{23} and D_{33}?

In order for the multiplication of two matrices to be defined, the matrices must be conformable, which means that the number of columns in the first matrix must equal the number of rows in the second matrix.

Notice, also, that the order of the product matrix is the number of rows of the first matrix by the number of columns of the second. This can be shown schematically as follows.

$$\underset{(3 \times 2)}{B} \times \underset{(2 \times 3)}{C} = \underset{(3 \times 3)}{D.}$$

Same

Dimensions of the product

In general, if we multiply a matrix P with m rows and k columns times a matrix Q with k rows and n columns, the products will be a matrix R with m rows and n columns.

$$\underset{(m \times k)}{P} \times \underset{(k \times n)}{Q} = \underset{(m \times n)}{R.}$$

Same

Dimensions of the product

The dimensions of these matrices can also be described using the row and column labels. Matrix B classifies the data according to Options (rows) and Foods (columns). Hence you can refer to matrix B as an Options by Foods matrix. Likewise you can describe C as a Foods by Houses matrix. The product B times C, in turn, results in a matrix of dimension Options by Houses, which is what you wanted to know. Notice, also, that when the multiplication was performed, the common label (Foods) was eliminated, so the product matrix was left with the row label of the first factor and the column label of the second. Schematically, the dimension of each of the matrices involved in computing the product can be described as follows.

B	$\times$	C	$=$	$D.$
Options by foods		Foods by houses		Options by houses

Same

Dimensions of the product

Using row and column labels in this manner helps determine whether a matrix multiplication will result in a meaningful interpretation or, indeed, whether it will give you the results that you wish.

Exercises

1. Mike, Liz, and Kate are heirs to an estate that consists of a condominium, a customized BMW, and choice season tickets to the Nebraska Cornhusker football games, and for the purposes of fair division, they have submitted the bids shown in matrix *E*.

$$E = \begin{array}{r} \\ \text{Mike} \\ \text{Liz} \\ \text{Kate} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Condo} & \text{BMW} & \text{Tickets} \end{array} \\ \begin{bmatrix} \$185{,}000 & \$76{,}000 & \$250 \\ \$175{,}000 & \$60{,}000 & \$215 \\ \$180{,}000 & \$75{,}000 & \$325 \end{bmatrix} \end{array}.$$

The awarding of the items in the estate is indicated by matrix *A*.

$$A = \begin{array}{r} \\ \text{Condo} \\ \text{BMW} \\ \text{Tickets} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Mike} & \text{Liz} & \text{Kate} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}.$$

a. Find the matrix product $P = EA$. Label the rows and columns of *P*.

b. Write an interpretation of the entries in matrix *P*. (Refer to Exercise 6 in Lesson 2.2, pages 56 and 57.)

2. Emma and Ken go out to eat at Sammy's Drive Inn. Ken has a small cheeseburger, a baked potato with sour cream, and a shake. Emma orders a Sammy's special, medium fries, and a shake. The approximate numbers of calories, grams of fat, and milligrams of

Industries such as the fast-food business have changed dramatically since the advent of the computer in using matrices as a natural tool for storing and manipulating data.

cholesterol in each of these foods are represented in the following table.

	Calories	Fat (g)	Cholesterol (mg)
Cheeseburger	450	40	50
Sammy's special	570	48	90
Potato/sour cream	500	45	25
French fries	300	30	0
Shake	400	22	50

a. Write a matrix Q that describes Emma's and Ken's choices, with the columns' representing the foods. Label the rows and columns of this matrix.
b. Write a matrix C that represents the information in the preceding table with the rows' representing the foods. Label the rows and columns of this matrix.
c. What is the dimension of matrix Q and of matrix C?
d. What is the dimension of the product Q times C? Show why your answer is correct by using a schematic diagram.
e. The dimension of matrix Q could be described as Persons by Foods. Describe the dimensions of matrices C and Q times C in a similar manner. Justify your answer for matrix Q times C with a schematic diagram.
f. Multiply matrix Q times matrix C to get a matrix R. Label the rows and columns of matrix R.
g. Interpret R_{12}, R_{21}, and R_{23}.

3. a. What must be true about the dimensions of matrices A and B if the product $C = AB$ is defined?
b. If the products AB and BA are both defined, what must be true about the dimensions of matrices A and B? Why?
c. Find two nonsquare matrices A and B, where AB and BA are both defined. Compute AB and BA. Does $AB = BA$? Why?
d. As illustrated by your answer in part b, if AB and BA are both defined, it does not necessarily follow that $AB = BA$ (i.e., *in general, matrix multiplication is not commutative*). Using 2×2 matrices, find examples in which $AB = BA$ and in which AB is not equal to BA.

4. Let A be any 3×3 matrix and let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that $IA = AI = A$.

The matrix I is called an **identity matrix.** An identity matrix is any matrix in which each of the entries along the main diagonal are 1s and all other entries are 0s. Identity matrices act in the same way for matrix products as the number 1 does for number products.

5. Given the matrices A, B, and C.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

a. Do you think that $A(BC) = (AB)C$?
b. Test your conjecture by computing the products $A(BC)$ and $(AB)C$.
c. The computations in part b show one case in which **matrix multiplication is associative.** Do you think this property holds for all matrices A, B, and C for which the product $A(BC)$ is defined? Why or why not?

6. Find two (2×2) matrices A and B to demonstrate that $(A + B)(A - B)$ is not necessarily equal to $A^2 - B^2$.

7. In algebra you learned that two numbers whose product is 1 (the identity element for multiplication) are called inverses of each other. For example, 5 and $\frac{1}{5}$ (or 5^{-1}) are inverses of each other since $5(\frac{1}{5}) = (\frac{1}{5})5 = 1$.

Similarly, if A and B are two square matrices such that $AB = BA = I$, then A and B are called **inverses** of each other. The inverse of A is denoted A^{-1}.

a. Verify that the matrices A and B are inverses of each other by computing AB and BA.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

b. Not all square matrices will have an inverse. Use algebra to show that matrix C does not have an inverse.

$$C = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

8. Carefully plot the points $A(0, 0)$, $B(6, 2)$, $C(8, 6)$, and $D(2, 4)$ on graph paper. Connect the points to form a polygon $ABCD$. This polygon can be represented with a matrix P as follows.

$$P = \begin{matrix} \; A & B & C & D \\ \end{matrix}$$
$$P = \begin{bmatrix} 0 & 6 & 8 & 2 \\ 0 & 2 & 6 & 4 \end{bmatrix}.$$

a. Multiply the matrix representing polygon $ABCD$ by the matrix

$$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

b. Plot and label the four points represented in your new matrix as A', B', C', and D'. Connect the points to form polygon $A'B'C'D'$.

c. Describe the relationship between polygon $A'B'C'D'$ and polygon $ABCD$.

d. Multiply the matrix representing polygon $A'B'C'D'$ by the matrix

$$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

e. Plot and label the four points represented in your new matrix as A'', B'', C'', and D''. Describe the relationship between polygon $A''B''C''D''$, and polygon $A'B'C'D'$.

f. Multiply T_2T_1 to get a new matrix R. Multiply R times the matrix P, that represents the original polygon $ABCD$, and plot the resulting points. What effect does multiplication by R have on $ABCD$? Do the following to test your conjecture: Use a blank sheet of unlined paper and trace both your axes and polygon $ABCD$. Leave your copy on top of the original polygon and place the point of your pencil on the origin. Now, holding the original paper in place, rotate the top sheet until your copy of $ABCD$ rests on top of polygon $A''B''C''D''$. Describe what happened to polygon $ABCD$.

g. Find a matrix T_3 that reflects polygon $A''B''C''D''$ about the y-axis into quadrant IV of your graph.

h. Find a matrix T_4 that rotates polygon $A'B'C'D'$ about the origin into quadrant IV. How does T_4 relate to T_2 and T_3?

For Exercises 9, 10, and 11, you need either a graphing calculator or access to computer software that performs matrix operations.

9. The matrix A is called an upper-triangular matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

a. Calculate A^2, A^3, and A^4.
b. Make a conjecture about the form of A^k.
c. Test your conjecture by computing additional powers of A.
d. Challenge: Prove your conjecture using mathematical induction.

10. The Fancy Bag manufacturing company that makes and markets fine leather bags has three factories—one in New York, one in Nebraska, and one in California. One of the bags they make comes in three styles—handbag, standard shoulder bag, and roomy shoulder bag. The production of each bag requires three kinds of work—cutting the leather, stitching the bag, and finishing the bag.

Matrix T gives the time (in hours) of each type of work required to make each type bag.

$T =$

	Cutting	Stitching	Finishing
Handbag	0.4	0.6	0.4
Standard	0.5	0.8	0.5
Roomy	0.6	1.0	0.6

Matrix P gives daily production capacity at each of the factories.

$P =$

	Handbag	Standard	Roomy
New York	10	15	20
Nebraska	25	15	12
California	20	12	10

Matrix W provides the hourly wages of the different workers at each factory.

$W =$

	Cutting	Stitching	Finishing
New York	7.50	8.50	9.00
Nebraska	7.00	8.00	8.50
California	8.40	9.60	10.10

Matrix D contains the total orders received at each factory for the months of May and June.

$$D = \begin{array}{l} \text{Handbag} \\ \text{Standard} \\ \text{Roomy} \end{array} \begin{array}{c} \begin{array}{cc} \text{May} & \text{June} \end{array} \\ \begin{bmatrix} 600 & 800 \\ 800 & 1{,}000 \\ 400 & 600 \end{bmatrix} \end{array}.$$

a. Matrix T can be described as a Bag by Work matrix. Describe matrices P, W, and D in a similar manner.
   Use the matrices above (or their transposes) to compute the following. Label the rows and columns of the matrix in each answer. Hint: The label dimensions from part a will help you decide what your matrix products should look like.
b. The hours of each type of work needed each month to fill all orders.
c. The production cost per bag at each factory.
d. The cost of filling all May orders at the Nebraska factory. (Hint: In this example the answer, a single value, is the product of a row vector and a column vector).
e. The daily hours of each type of work needed at each factory if production levels are at capacity.

11. (For those students who have studied trigonometry.)
a. Plot the polygon $ABCD$ represented in Exercise 8.
b. Multiply the matrix P by the following transformation matrix.

$$T_1 = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}.$$

c. Plot the resulting polygon and label it $A'B'C'D'$. How does polygon $A'B'C'D'$ relate to polygon $ABCD$? Try repeating the transformation using 180° to test your conjecture.
d. Write a matrix that will rotate a polygon through 60°. Does this transformation matrix have the same effect as applying T_1 twice? Test your conjecture.
e. Find a matrix that rotates polygon $ABCD$ through 90° and another that rotates it through −90°. Find the product of these two transformation matrices. What is the relationship between these two matrices? Test your conjecture by finding the product of the matrices that will rotate the polygon through 60° and −60°.

12. Challenge: Refer to Exercise 9 and explore the following.

a. Replace the 1s in the upper-triangular matrix A with 2s, 3s, and 4s and repeat parts a to d of Exercise 9 for each of your new upper-triangular matrices.

b. Use the results of part a to make a conjecture for A^k when the 1s in A are replaced by any natural number m.

c. Prove your conjecture in part b using mathematical induction.

Computer/Calculator Exploration

13. Write a program for the graphing calculator based on the method of Exercise 11 that will allow you to enter the coordinates of the vertices of a polygon and the angle of rotation. Design your program so that both the original polygon and the rotation will be displayed.

Project

14. Research and write a short report on additional applications of matrices in trigonometry. Possible topics include the representation vectors and complex numbers as matrices.

Lesson 3.4

Population Growth: The Leslie Model, Part 1

Age-specific population growth is a topic that is of great concern to people in fields as diverse as urban planning and wildlife management. Urban policymakers are interested in knowing how many people there will be in various age groups after certain periods of time have elapsed. Those in wildlife management worry about maintaining animal populations at levels that can be supported in their natural habitats without damage to the environment.

If the age distribution of a population at a certain date is known, along with birth and survival rates for age-specific groups, a model can be created to determine the age distributions of the survivors and descendants of the original population at successive intervals of time. The problem used to illustrate this model was posed in 1945 by P. H. Leslie of the Bureau of Animal Population at Oxford University in Oxford, England. In this problem, the growth rate of a population of an imaginary species of small brown rats, *Rattus norvegicus,* is examined. The lifespan of these rodents is 15–18 months. They have their first litter at approximately 3 months and continue to reproduce every 3 months until they reach the age of 15 months. Birthrates and age-specific survival rates for 3-month periods are summarized in the following table. In order to simplify the situation as much as possible, birth rates and survival rates are held constant over time and only the female population is considered.

Age (months)	Birthrate	Survival Rate
0–3	0	0.6
3–6	0.3	0.9
6–9	0.8	0.9
9–12	0.7	0.8
12–15	0.4	0.6
15–18	0	0

The actual number of female births in a particular age group can be found by multiplying the birth rate by the number of females currently in the age group. The survival rate is the probability that a rat will survive and move into the next age group.

Suppose the original female rat population is 42 animals with the age distribution shown in the following table.

Age (months)	0–3	3–6	6–9	9–12	12–15	15–18
Number	15	9	13	5	0	0

In examining this model, one question that might be asked is how many female rats will there be after 3 months have passed. Another question might be asked about the age distribution of this new group. To answer these questions, it is necessary to find the number of new female babies introduced into the population and the number of female rats that survive in each group and move up to the next age group.

The number of new births after 3 months (1 cycle) can be found by multiplying the number of female rats in each age group times the corresponding birth rates and then finding the sum:

$$15(0) + 9(0.3) + 13(0.8) + 5(0.7) + 0(0.4) + 0(0)$$
$$= 0 + 2.7 + 10.4 + 3.5 + 0 + 0 = 16.6.$$

The number of female rats in the 0–3 age group after 3 months is about 17. The number of female rats who survive in each age group and move up to the next can be found as follows (SR stands for survival rate).

Age	No.	SR	Number moving up to the next age group
0–3	15	0.6	(15)(0.6) = 9.0 move up to the 3–6 age group.
3–6	9	0.9	(9)(0.9) = 8.1 move up to the 6–9 age group.
6–9	13	0.9	(13)(0.9) = 11.7 move up to the 9–12 age group.
9–12	5	0.8	(5)(0.8) = 4.0 move up to the 12–15 age group.
12–15	0	0.6	(0)(0.6) = 0 move up to the 15–18 age group.
15–18	0	0	No rodent lives beyond 18 months.

The sum of the number of female rats in each age group results in a total population of female rats equal to 16.6 + 9.0 + 8.1 + 11.7 + 4.0 + 0, or 49.4.

After 3 months (1 cycle) the female rat population has grown from 42 to approximately 50. The distribution of female rats is shown in the following table. Notice that in the table the number of female rats in each age group is not rounded to the nearest integer. This is because when the values are to be used for further analysis, rounding off can mean a significant difference in calculations over time even though it doesn't make sense to have a fractional part of a rat.

Age	0–3	3–6	6–9	9–12	12–15	15–18
Number	16.6	9.0	8.1	11.7	4.0	0

Exercises

1. Use the preceding table (the distribution of the female rat population after 3 months) and the process introduced in this lesson to compute the following. Note that each of the calculations refers to female rats only.

a. Calculate the number of newborn rats (aged 0–3) after 6 months (2 cycles).

b. Calculate the number of rats that survive in each age group after 6 months and move up to the next age group.

c. Use the results to parts a and b to show the distribution of the rat population after 6 months. Approximately how many rats will there be after 6 months?

d. Use your population distribution from part c to calculate the number of rats and the approximate number in each age group after 9 months (3 cycles). Continue this process to find the number of rats after 12 months (4 cycles).

e. Compare the original number of rats with the numbers of rats after 3, 6, 9, and 12 months. What do you observe?
f. What do you think might happen to this population if you extended the calculations to 15, 18, 21, . . . months?

2. Suppose that a species of deer has the following birth and survival rates.

Age (years)	Birthrate	Survival Rate
0–2	0	0.6
2–4	0.8	0.8
4–6	1.7	0.9
6–8	1.7	0.9
8–10	0.8	0.7
10–12	0.4	0

a. Given that the initial population for this species is 148 deer with the following distribution,

Age (years)	0–2	2–4	4–6	6–8	8–10	10–12
Number	50	30	24	24	12	8

find the number of newborn female deer after 2 years (1 cycle).
b. Arrange the initial population distribution in a row matrix and the birth rates in a column matrix. Multiply the row matrix times the column matrix. Interpret this result.
c. Calculate the number of deer that survive in each age group after 2 years and move up to the next age group.
d. Explore the possibility of multiplying the initial population distribution in a row matrix times some column matrix to find the number of deer after 2 years that move from:
   i. The 0–2 group to the 2–4 group. (Hint: the column matrix that you use will need to contain several zeros in order to produce the desired product.)
   ii. The 2–4 to the 4–6 group.
   iii. The 4–6 group to the 6–8 group.
   iv. The 6–8 group to the 8–10 group.
   v. The 8–10 group to the 10–12 group.

3. Using the birth and survival rate information for *Rattus norvegicus* from this lesson (see the table on page 133), find the population total and

distribution after 3 months (1 cycle) for the following initial populations.
a. [35 0 0 0 0 0].
b. [5 5 5 5 5 5].

4. Using the birth and survival rate information for the deer population in Exercise 2, find the population total and distribution after 2 years (1 cycle), 4 years (2 cycles), 6 years (3 cycles), 8 years (4 cycles), and 10 years (5 cycles) if the initial population is [25 0 0 0 0 0].

Population Growth: The Leslie Model, Part 2

In your beginning explorations of the Leslie model for population growth, you found that it was possible to use an initial population distribution, birth rates, and survival rates to predict population figures at future times. Looking 2, 3, or even 4 cycles into the future is not impossible, but the arithmetic soon becomes cumbersome. What do the wildlife manager and the urban planner do if they want to look 10, 20, or even more cycles into the future?

In Lesson 3.4, Exercise 2, you began to get a glimpse of the model that P. H. Leslie proposed. The use of matrices seems to hold the key, and with the aid of computer software or a calculator, looking ahead many cycles is not difficult. In fact, some very fascinating results are produced.

Let's return to our rat model. If the original population distribution (P_0) and a matrix that we will call L are multiplied, the population distribution at the end of cycle 1 (P_1) can be calculated.

$$P_0L = [15 \quad 9 \quad 13 \quad 5 \quad 0 \quad 0]\begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.9 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0.9 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= [15(0) + 9(0.3) + 13(0.8) + 5(0.7) + 0(0.4) + 0(0) \quad 15(0.6) \quad 9(0.9) \quad 13(0.9) \quad 5(0.8) \quad 0(0.6)]$$

$$= [16.6 \quad 9.0 \quad 8.1 \quad 11.7 \quad 4.0 \quad 0] = P_1.$$

The matrix L (called the **Leslie matrix**) is formed by augmenting or joining the column vector containing the birth rates of each age group and a series of column vectors that contain a survival rate as one entry and zeros everywhere else. Notice that the survival rates (of which there are $n - 1$ since no animal survives beyond the 15–18 age group) lie along the **super diagonal** that is immediately above the main diagonal of the matrix.

When the matrix L is multiplied by a population distribution P_k, a new population distribution P_{k+1} results. To find population distributions at the end of other cycles, the process can be continued.

$$P_1 = P_0L$$

$$P_2 = P_1L = (P_0L)L = P_0(LL) = P_0L^2.$$

In general, $P_k = P_0L^k$.

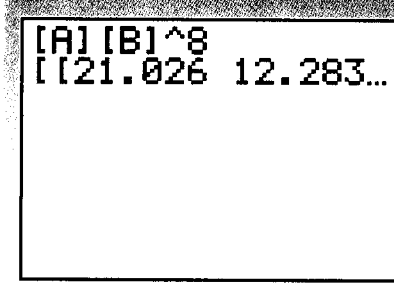
Using this formula to find the population distribution for the rats after 24 months (8 cycles) and the total population of the rats, you have

$$P^8 = P_0L^8 = [15 \quad 9 \quad 13 \quad 5 \quad 0 \quad 0]\begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.9 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0.9 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^8$$

$$= [21.03 \quad 12.28 \quad 10.90 \quad 9.46 \quad 7.01 \quad 4.27].$$

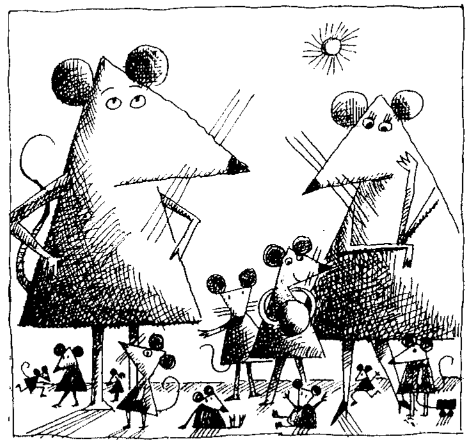
Total population = 21.03 + 12.28 + 10.90 + 9.46 + 7.01 + 4.27 = 64.95, or approximately 65 rats.

Point of Interest



You can perform the calculation P_0L^8 on a calculator with matrix features.

Exercises



Note: For the following exercises, you need to have access to either a graphing calculator or computer software that performs matrix operations.

1. Using the original population distribution, [15 9 13 5 0 0], and Leslie matrix from the *Rattus norvegicus* example,
   a. Find the population distribution after 15 months (5 cycles).
   b. Find the total population after 15 months. (Hint: Multiply P_0L^5 times a column matrix consisting of six 1s.)
   c. Find the population distribution and the total population after 21 months.

2. Suppose the *Rattus norvegicus* start dying off from overcrowding when the total female population for a colony reaches 250. Find how long it will take for this to happen when the initial population distribution is:
   a. [18 9 7 0 0 0].
   b. [35 0 0 0 0 0].
   c. [5 5 5 5 5 5].
   d. [25 15 10 11 7 13].

3. a. Complete the table for the given cycles of *Rattus norvegicus* using the original population distribution of [15 9 13 5 0 0].

Cycle	Total Population	Growth Rate
Original	42	
1	49.4	17.6%
2	56.08	13.5%
3	57.40	2.4%
4		
5		
6		

   b. What do you observe about the growth rates?
   c. Calculate the total populations for P_{25}, P_{26}, and P_{27}. What is the growth rate between these successive years? Hint: To find the growth

rate from P_{25} to P_{26}, subtract the total population for P_{25} from the total population for P_{26} and divide the result by the total population for P_{25}.

4. One characteristic of the Leslie model is that growth does stabilize at a rate called the **long-term growth rate** of the population. As you observed in Exercise 4, the growth rate of *Rattus norvegicus* converges to about 3.04%. This means that for a large enough k, the total population in cycle k will equal about 1.0304 times the total population in the previous cycle.

a. Find the long-term growth rate of the total population for each of the initial population distributions in Exercise 2.

b. How does the initial population distribution seem to affect the long-term growth rate?

5. Consider once again the deer species from Exercise 2 in Lesson 3.4. The birth and survival rates follow.

Age (years)	Birthrate	Survival Rate
0–2	0	0.6
2–4	0.8	0.8
4–6	1.7	0.9
6–8	1.7	0.9
8–10	0.8	0.7
10–12	0.4	0

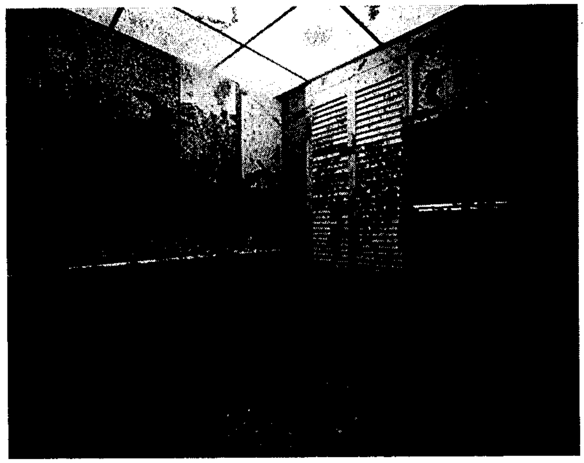
a. Construct the Leslie matrix for this animal.

b. Given that $P_0 = [50\ 30\ 24\ 24\ 12\ 8]$, find the long-term growth rate.

c. Suppose the natural range for this animal can sustain a herd that contains a maximum of 1,250 females. How long before this herd size is reached?

d. Once the long-term growth rate of the deer population is reached, how might the population of the herd be kept constant?

6. In his study of the application of matrices to population growth, P. H. Leslie was particularly interested in the special case in which the birthrate vector has only one nonzero element. The following example falls into this special case. Suppose there is a certain kind of bug that lives at most 3 weeks and reproduces only in the third week of life. Fifty percent of the bugs born in one week survive into the second week, and 70% of the bugs who survive into their second week also survive into



their third week. On the average, six new bugs are produced for each bug that survives into its third week. A group of five 3-week-old female bugs decide to make their home in a storage box in your basement.

a. Construct the Leslie matrix for this bug.
b. What is P_0?
c. How long will it be before there are at least 1,000 female bugs living in your basement?

7. Exercise 6 is an example of a population that grows in waves. Will the population growth for this population stabilize in any way over the long run? To explore this question, make a table of the population distributions P_{22} through P_{30}.

a. Examine the population change from one cycle to the next. Can you find a pattern in the population growth?
b. Examine the population change from P_{22} to P_{25}, P_{23} to P_{26}, P_{24} to P_{27}, P_{25} to P_{28}, P_{26} to P_{29}, and P_{27} to P_{30}. Are you surprised at the results? Why?

8. Change the initial population in Exercise 6 to $P_0 = [4\ 4\ 4]$ and repeat the instructions in Exercise 7 looking at the total population growth for each cycle.

9. Examine the changes in successive age groups from P_{22} to P_{25}, P_{23} to P_{26}, P_{24} to P_{27}, P_{25} to P_{28}, P_{26} to P_{29}, and P_{27} to P_{30}. Make a conjecture based on your results.

10. Using mathematical induction, prove that $P_k = P_0L^k$ for any original population P_0 and Leslie matrix L, where k is a natural number.

Projects

11. Search the Web for applications of the Leslie matrix model in managing wildlife or domestic herds.

12. Research and report on the life and work of P. H. Leslie.

Chapter Extension

Harvesting Animal Populations

The following application of the Leslie matrix to a population of domestic sheep in New Zealand was originally published in 1967. (See G. Caughley, "Parameters for Seasonally Breeding Populations," *Ecology* 48(1967):834–839. Anton and Rorres (1987) developed the problem in their text, *Elementary Linear Algebra with Applications.* More recent references can be found by searching the Web for Leslie matrix applications.

This application involves what is referred to as the *harvesting* of a population. The term *harvesting* is defined as removal of the animals from the population. This could entail slaughtering some of the animals as is the case of wild deer or caribou herds that grow too large to be supported by their habitat. It could also mean selling or relocating some of the animals to start a new herd or colony as is the case of domestic herds or wild colonies of animals such as beaver. The ultimate goal is to find a stable distribution from which the population growth can be harvested at regular intervals so that the population can be held constant.

This model begins with an initial population that undergoes a growth period that is described by the Leslie matrix. At the end of the growth period, however, a certain percentage of each age group in the distribution is harvested. This is usually done in such a way that the unharvested population has the same age distribution as the initial population. A plan for harvesting the same percentage of each age group on a regular basis so the population remains the same after each harvesting is called a *sustainable harvesting policy.*

To describe this situation using matrices, suppose a Leslie matrix and a population distribution vector for n age groups of females in a population have been defined. If h_i, for $i = 1, 2, 3, \ldots, n$, represents the fraction of females harvested for each of the age groups, an $n \times n$ diagonal matrix H called the harvesting matrix can be formed as follows.

$$H = \begin{bmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & h_n \end{bmatrix}.$$

If all the h_i's are the same value, then the harvesting is called uniform. From the definition of a sustainable harvesting policy, it follows that

$$\begin{bmatrix} \text{Age distribution} \\ \text{at end of} \\ \text{growth period} \end{bmatrix} - [\text{harvest}] = \begin{bmatrix} \text{age distribution} \\ \text{at beginning of} \\ \text{growth period} \end{bmatrix}.$$

Using matrix notation, where P represents the distribution at the beginning of the growth period, L represents the Leslie matrix, and H represents the harvesting matrix, this translates to

$$PL - PLH = P.$$

This model was applied in 1967 to a species of domestic sheep in New Zealand which has a lifespan of 12 years. The sheep were divided into 12 age groups with a growth period of 1 year. Birth and survival rates for each age group of sheep were found using demographics, and the following Leslie matrix was developed.

$$\begin{bmatrix}
0 & 0.845 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.045 & 0 & 0.975 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.391 & 0 & 0 & 0.965 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.472 & 0 & 0 & 0 & 0.950 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.484 & 0 & 0 & 0 & 0 & 0.926 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.546 & 0 & 0 & 0 & 0 & 0 & 0.895 & 0 & 0 & 0 & 0 & 0 \\
0.543 & 0 & 0 & 0 & 0 & 0 & 0 & 0.850 & 0 & 0 & 0 & 0 \\
0.502 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.786 & 0 & 0 & 0 \\
0.468 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.691 & 0 & 0 \\
0.459 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.561 & 0 \\
0.433 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.370 \\
0.421 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Snow Goose Glut: Hunters Will Get Every Advantage

LINCOLN JOURNAL STAR
January 23, 1999

When snow geese arrive in south central Nebraska in the coming weeks, waterfowl hunters will be ready for them. They'll be allowed to kill 20 snow geese per day and no conservation officer will ever ask to check their possession limits, because there are no possession limits.

Wildlife managers are trying to address a staggering problem of "white" or "light" geese population. Biologists estimate that 3 million light geese will migrate through Nebraska and they say the number should be reduced to about 1 million. As many as a million birds have been reported on a single 2,000-acre wetland in Nebraska's Rainwater Basin.

By grubbing their beaks into fragile tundra soils, the huge numbers of geese are causing long-term damage to the summer arctic habitat used by many other waterfowl and shorebirds. This is the reason the Nebraska Game and Parks Commission has taken such dramatic steps to help hunters reduce the population.

If left to reproduce without a harvesting policy in place, the sheep would eventually approach a stable growth rate. If the shepard allows this to happen without a harvesting policy, the income from selling the wool would not cover the cost of feeding the flock. The stable growth rate that can be found using the Leslie matrix can be used to approximate a uniform harvesting policy. In this case the uniform harvesting policy is one in which roughly 18% of the sheep from each of the 12 age groups is harvested each year.

Chapter 3 Review

1. Write a summary of what you think are the important points of this chapter.

2. Can a matrix with dimension 3 by 5 be added to a matrix with dimension 5 by 3? Explain your answer.

3. The math club is planning a Saturday practice session for an upcoming math contest. For lunch they have ordered 35 Mexican special combination lunches, 6 large bags of corn chips, 6 containers of salsa sauce, and 12 six-packs of assorted cold drinks.
   a. Write this information in a row matrix L. Label your matrix.
   b. Interpret L_2 and L_4.
   c. The math club pays \$4.50 per lunch, \$1.97 per bag of corn chips, \$2.10 for each container of salsa, and \$2.89 for each six-pack of cold drinks. Use multiplication of a row and column vector to find the total cost to the math club. Label your vectors.

4. A youth fellowship group is planning a spring retreat. They have contacted three lodges in the vicinity to inquire about rates. They found that Crystal Lodge charges \$13.00 per person per day for lodging, \$20.00 per day for food, and \$5.00 per person for use of the recreational facilities. Springs Lodge charges \$12.50 for lodging, \$19.50 for meals, and \$7.50 for use of the recreational facilities. Bear Lodge charges \$20.00 per night for lodging, \$18.00 a day for meals, and there is no extra charge for using the recreational facilities. Beaver Lodge charges a flat rate of \$40.00 a day for lodging (meals included) and no additional fee for use of the recreational facilities.
   a. Display this information in a matrix C. Label the rows and columns.
   b. State the values of C_{22} and C_{43}
   c. Interpret C_{13} and C_{31}.

5. Mrs. Jones has been bothered by flies, spiders, and a variety of beetles on her summer porch. She has been shopping for a vacuum-powered insect disposal system. She found one at Z-Mart and another model at Base Hardware. The Z-Mart system cost $39.50, disposal cartridges were 6 for $24.50, and storage cases were $8.50 each. At Base Hardware the system cost $49.90, cartridges were 6 for $29.95, and cases were $12.50 each.

a. Write and label a matrix showing the prices for the three parts at the two stores.
b. Mrs. Jones decided to wait and see if the prices for insect disposal systems would be reduced during the upcoming end-of-summer sales. When she went back during the sales, the Z-Mart prices were reduced by 10% and the Base Hardware prices were reduced by 20%. Construct a matrix showing the sale prices for each of the three parts at the two stores.
c. Use matrix subtraction to compute how much Mrs. Jones could save for each part at the two stores.
d. Suppose Mrs. Jones is interested in purchasing the vacuum-powered insect disposal systems for herself and three of her friends. Use multiplication of a matrix by a scalar to show how much she would pay for four of each part of the system at the two stores at the sale prices.

6. An artist fashions plates and bowls from small pieces of colored woods. The artist currently has orders for five 10-inch plates, three large bowls, and seven small bowls. Each plate requires 100 pieces of ebony, 800 pieces of walnut, 600 pieces of rosewood, and 400 pieces of maple. It takes 200 ebony pieces, 1,200 walnut pieces, 1,000 rosewood pieces, and 800 pieces of maple to make a large bowl. A small bowl takes 50 pieces of ebony, 500 walnut pieces, 450 rosewood pieces, and 400 pieces of maple.

a. Write a row matrix showing the current orders for this artist's work.
b. Construct a matrix showing the number of pieces of wood used in an individual plate or bowl.
c. Use matrix multiplication to compute the number of pieces of each type of wood the artist will need for the plates and bowls that are on order.
d. Suppose it takes the artist 3 weeks to fashion a plate, 4 weeks to make a large bowl, and 2 weeks to complete a small bowl. Use matrix multiplication to show how long it will take the artist to fill all the orders for plates and bowls.

7. Jon has money invested in three sports complexes in Smith City. His return (annual) from a $50,000 investment in a tennis club is 8.2%. He receives 6.5% from a $100,000 investment in a golf club and 7.5% on a $75,000 investment in a soccer club. Use vector multiplication to find Jon's income from his investments for one year. Label your vectors.

8. Three music classes at Central High are selling candy as a fund-raiser. The number of each kind of candy sold by each of the three classes so far is shown in the following table.

	Jazz Band	Symphonic Band	Orchestra
Chocolate delights	300	220	250
Chocolate overdose	240	330	400
Chocolate chewies	150	200	180
Sour balls	175	150	160

The profit for each type of candy is sour balls, 30 cents; chocolate overdose, 50 cents; chocolate delights, 25 cents; and chocolate chewies, 35 cents. Use matrix multiplication to compute the profit made by each class on its candy sales.

9. Write the transpose (A^T) of matrix A, where

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 1 & 3 \end{bmatrix}.$$

10. The dimensions of matrices P, Q, R, and S are 3×2, 3×3, 4×3, and 2×3, respectively. If matrix multiplication is possible, find the dimensions of the following matrix products. If it is not possible, tell why.
    a. QP.
    b. RQ.
    c. QS.
    d. RPS.

11. Let matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

    a. Calculate M^2, M^3, and M^4.
    b. Predict the components of M^5 and check your prediction.

c. Generalize to M^n, where n is a natural number.
d. Prove your conjecture in part c using mathematical induction.
e. Repeat parts a, b, c, and d for the matrix

$$M = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

12. Complete the following statement: If a square matrix A has an inverse A^{-1}, then the product AA^{-1} = the ______________ matrix I, where I is a ______________.

13. Which of the following matrices are inverses of each other? Explain your answers.

a. $\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$ and $\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$.

b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

c. $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -1 & 2 \end{bmatrix}$.

14. The students at Central High are planning to hire a band for the Senior Prom. Their choices are bands A, B, and C. They survey the Sophomore, Junior, and Senior classes and find the following percentages of students (regardless of sex) prefer the bands,

	10th	11th	12th
A	20%	35%	40%
B	30%	30%	25%
C	50%	35%	35%

The student population by class and sex is:

	Male	Female
10th	235	225
11th	205	215
12th	175	190

Use matrix multiplication to find:
a. The number of males and females who prefer each band.
b. The total number of students who prefer each band.

15. The characteristics of the female population of a herd of small mammals is shown in the following table.

	Age Groups (months)					
	0–4	4–8	8–12	12–16	16–20	20–24
Birthrate	0	0.5	1.1	0.9	0.4	0
Survival Rate	0.6	0.8	0.9	0.8	0.6	0

Suppose the initial female population for the herd is given by

$$P_0 = [22 \quad 22 \quad 18 \quad 20 \quad 7 \quad 2].$$

a. What is the expected life span of this mammal?
b. Construct the Leslie matrix for this population.
c. Determine the long-term growth rate for the herd.
d. Suppose this mammal starts dying off from overcrowding when the total female population for the herd reaches 520. How long will it take for this to happen?

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