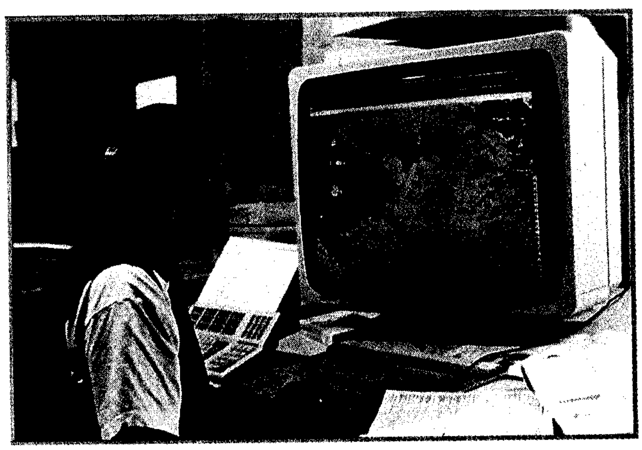
CHAPTER

Graphs and Their Applications

4

When the boundaries or names of countries change, cartographers have to be prepared to provide the public with new maps. For years, mapmakers and mathematicians alike have wondered about the number of colors it takes to color a map.

What is the minimum number of colors needed to color any map? How do you optimally color a map? What do map coloring and meeting time scheduling for your school organizations have in common? Whether you're a cartographer who must find a way to color a map, a businessperson who must determine whether a project can be completed on time, or a planner who wants to know the most efficient way to route a city's garbage trucks, you'll find the answer in an area of mathematics known as graph theory.



Lesson 4.1

Modeling Projects

How does a building contractor organize all of the jobs needed to complete a project? How do your parents manage to get all the food for a Thanksgiving dinner done at the same time? Many people believe that planning is a simple activity. After all, everybody does it. Planning your day-to-day activities seems to be second nature. What most people fail to realize is that for people in the business world who must plan and control work on extensive projects, this haphazard manner of planning is not the most efficient way to complete a job. A more scientific, organized method must be used.

One way to model projects that consist of several different subprojects, or tasks, is through the use of a diagram, or **graph,** that is made up of points called **vertices** and connecting lines called **edges** (See Figure 4.1).

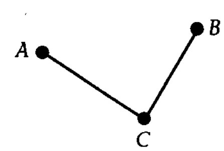


Figure 4.1 Graph with three vertices and two edges.

Explore This

The Central High yearbook staff has only 16 days left before the deadline for completing their book. They are running behind schedule and still have several tasks left to finish. The remaining tasks and time that it takes to complete each task are listed in the following table.

Task	Time (days)
Start	0
A Buy film	1
B Load cameras	1
C Take photos of clubs	3
D Take sports photos	2
E Take photos of teachers	1
F Develop film	2
G Design layout	5
H Print and mail pages	3

Is it possible for the yearbook to be completed on time if the tasks have to be done one after the other? If some tasks can be done at the same time as others, can the deadline be met?

As you may have noticed, some of the yearbook staff jobs can be done simultaneously, while several of them cannot be started until others have been completed. Assuming the following prerequisites, how soon can the project be completed?

Task	Time (days)	Prerequisite Task
Start	0	—
A Buy film	1	None
B Load cameras	1	A
C Take photos of clubs	3	*B*
D Take sports photos	2	*C*
E Take photos of teachers	1	*B*
F Develop film	2	*D*, *E*
G Design layout	5	*D*, *E*
H Print and mail pages	3	*G*, *F*

Drawing a graph of this information makes it easier to see the relationships among the tasks. In the graph in Figure 4.2 on page 154, the tasks are represented by points (vertices), and the arrows (directed edges) indicate which tasks must be finished before a new task can begin. Each edge also shows the number of days it takes to complete the preceding task. Note that tasks with the same prerequisites are aligned vertically. Although this is not necessary, it helps to make the graph easier to follow.

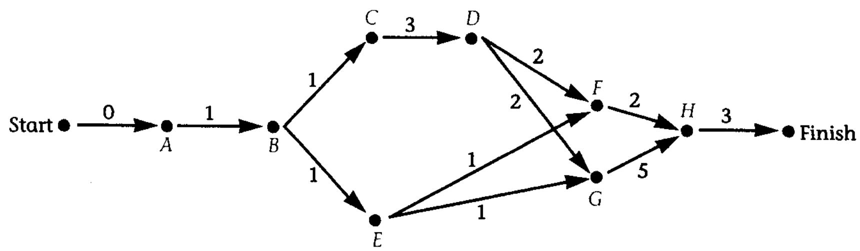


Figure 4.2 Diagram of the order of tasks necessary to complete the yearbook.

Exercises

In Exercises 1 through 4, use the task table to draw a graph with appropriately labeled vertices and edges.

1.

Task	Time	Prerequisites
Start	0	
A	5	None
B	6	*A*
C	4	*A*
D	4	*B*
E	8	*B, C*
F	4	*C*
G	10	*D, E, F*
Finish		

2.

Task	Time	Prerequisites
Start	0	
A	1	None
B	2	None
C	3	*A, B*
D	5	*B*
E	5	*C*
F	5	*C, D*
G	4	*D, E*
H	4	*E, F*
Finish		

3.

Task	Time	Prerequisites
Start	0	
A	5	None
B	7	*A*
C	4	*A*
D	3	*B*
E	7	*B, C*
F	5	*C*
G	8	*D, E, F*
Finish		

4.

Task	Time	Prerequisites
Start	0	
A	5	None
B	8	*A, D*
C	9	*B, I*
D	7	None
E	8	*B*
F	12	*I*
G	4	*C, E, F*
H	9	None
I	5	*D, H*
Finish		

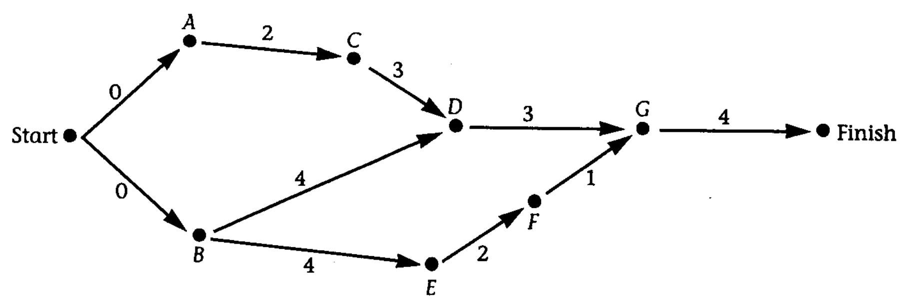
5. To help organize the task of completing the family dinner, Mrs. Shu listed the following tasks.

Task	Time (min.)	Prerequisite Task
Start	0	—
A Wash hands		
B Defrost hamburger		
C Shape meat into patties		
D Cook hamburgers		
E Peel and slice potatoes		
F Fry potatoes		
G Make salad		
H Set table		
I Serve food		

a. Complete the table by making reasonable time estimates in minutes for each of these tasks and indicating the prerequisites.
b. Construct a graph using the information from your table.
c. What is the least amount of time needed to prepare dinner?

6. Your best friend, Matt, has always been very disorganized. He is now preparing to leave for college and desperately needs your help.
a. Create a table of at least six activities that will need to be completed before Matt can leave home. Give the times and prerequisites of these activities.
b. Construct a corresponding graph.
c. For your task list, what is the least amount of time it will take to get Matt off to school?

7. Consider the following graph.



a. Complete the following task table for this graph.

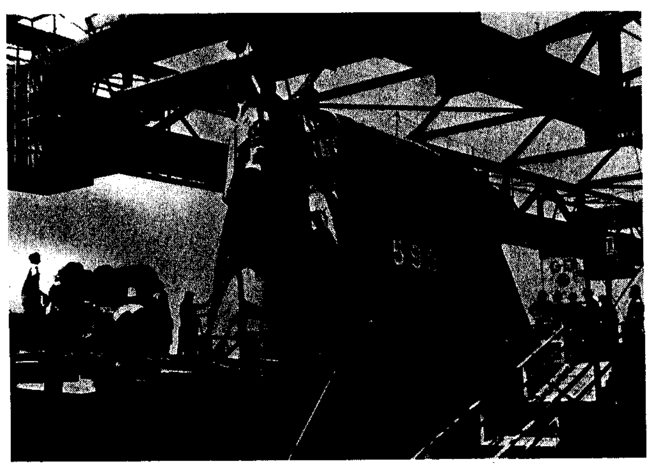
Task	Time	Prerequisite Task
Start	0	—
A		
B		
C		
D		
E		
F		
G		
Finish		

b. What is the least amount of time it will take to complete all of the tasks in this graph? Explain why it cannot be completed in less time.

Critical Paths

It is relatively easy to find the shortest time needed to complete a project if the project consists of only a few activities. But as the tasks increase in number, the problem becomes more difficult to solve by inspection alone.

In the 1950s the U.S. government was faced with the need to complete very complex systems such as the U.S. Navy Polaris Submarine project. In order to do this efficiently, a method was developed called PERT (Program Evaluation and Review Technique) in which those tasks which were critical to the earliest completion of the project were targeted. This path of targeted tasks from the start to the finish of the project became known as the **critical path**.



Recall the graph in Lesson 4.1 that represented the Central High yearbook project. How might you go about finding a systematic way to identify the critical path for this project? To do this, an **earliest-start time** (EST) for each task must be found. The EST is the earliest that an activity can begin if all the activities preceding it begin as early as possible.

To calculate the EST for each task, begin at the start and label each vertex with the smallest possible time that is needed before the task can begin. The label for *C* in Figure 4.3 is found by adding the EST of *B* to the 1 day that it takes to complete task *B* (1 + 1 = 2). Task *G* cannot be completed until both predecessors, *D* and *E*, have been completed. Hence, *G* cannot begin until 7 days have passed.

In the case of the yearbook staff, the earliest time in which the project can be completed is 15 days. As paradoxical as it may seem, the least amount of time that it takes to complete all of the tasks in the project corresponds to the time it takes to complete the longest path through the graph from start to finish. A path with this longest time is the desired critical path. In Figure 4.3, the critical path is Start–*ABCDGH*–Finish.

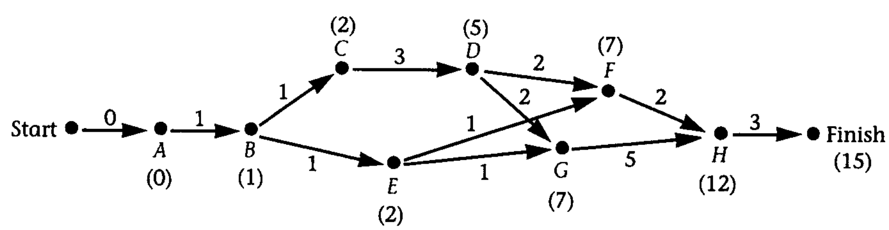
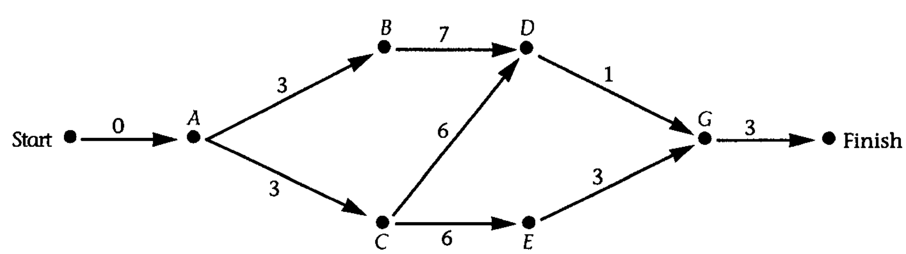


Figure 4.3 Yearbook diagram showing the earliest-start time for each task.

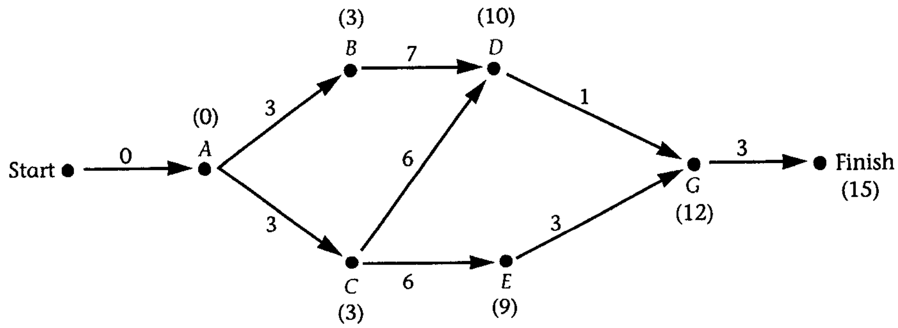
Example



1. Copy the graph, label the vertices with the EST for each task, and determine the earliest completion time for the project. All times are in minutes.
2. Identify the critical path.

The solutions are:

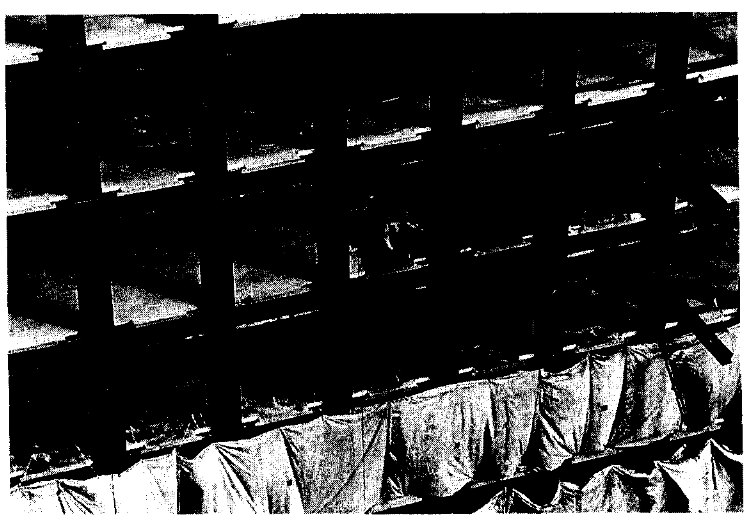
1.



The earliest time in which the project can be completed is 15 minutes.

2. Since the critical path is the longest path from the start to the finish, the critical path is Start–*ACEG*–Finish.

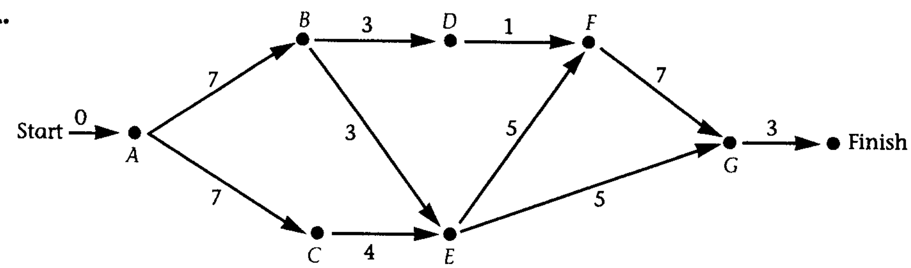
If it is desirable to cut the completion time of a project, it can be done by shortening the length of the critical path once it is found. In the preceding example, one way to shorten the time it takes to complete the project is to cut the time it takes to complete task *E* to 2 minutes instead of 3 minutes. If that is done, the completion time for the project is cut to 14 minutes.



The efficient management of large projects like the construction of a building requires the use of critical path analysis.

Exercises

1.



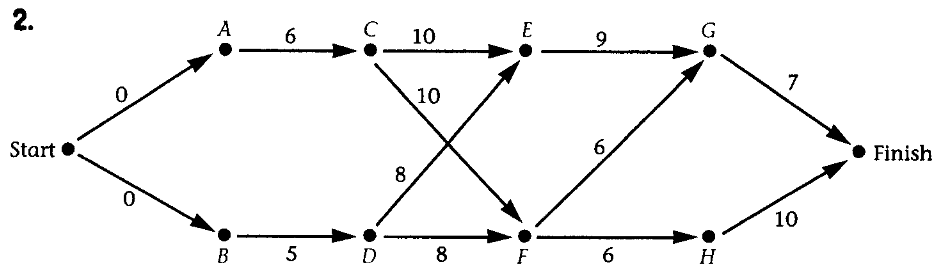
Complete the following.

Vertex	Earliest-Start Time
A	0
B	7
C	
D	
E	
F	
G	

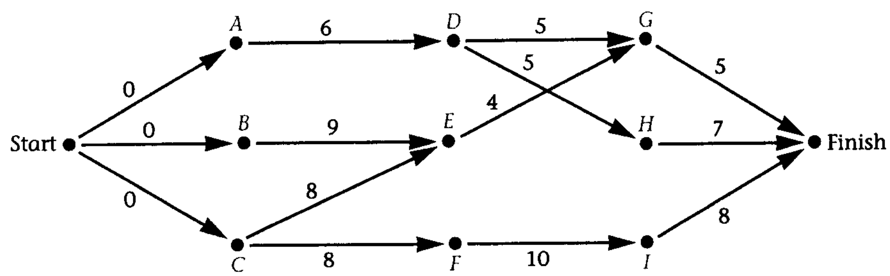
Minimum project time =
Critical path(s) =

In Exercises 2 and 3, list the vertices of the graphs and give their earliest-start time, as in Exercise 1. Determine the minimum project time and all of the critical paths.

2.



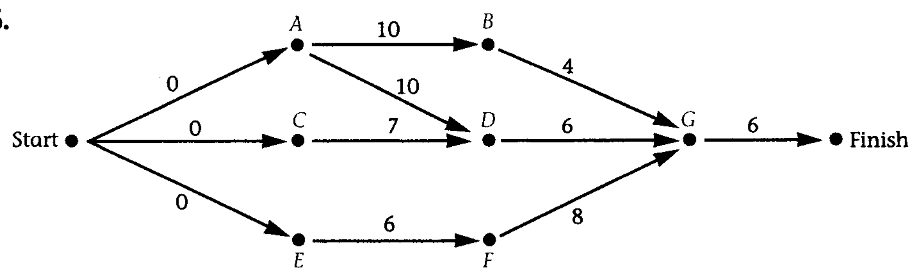
3.



4. Using the information from the following table, construct a graph and label each of the vertices with its earliest-start time. Determine the minimum project time and critical path.

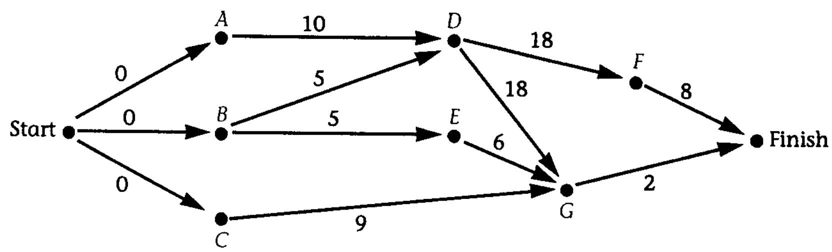
Task	Time	Prerequisites
Start	0	
A	2	None
B	4	None
C	3	*A*, *B*
D	1	*A*, *B*
E	5	*C*, *D*
F	6	*C*, *D*
G	7	*E*, *F*

5.

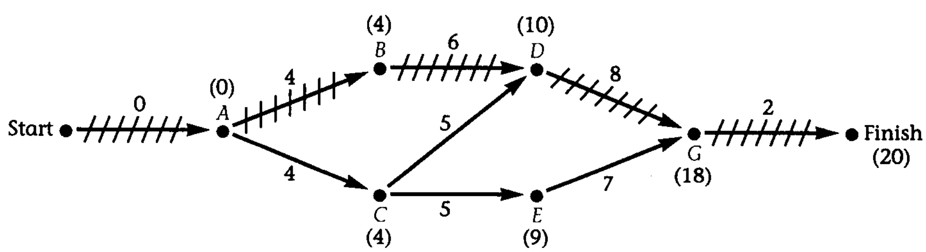


a. Copy the graph and label each vertex with its earliest-start time.
b. How quickly can the project be completed?
c. Determine the critical path.
d. What happens to the minimum project time if task *A*'s time is reduced to 9 days? To 8 days?
e. Will the project time continue to be affected by reducing the time of task *A*? Explain why or why not.

6. Construct a graph with three critical paths.
7. Determine the minimum project time and the critical path for the following graph.



8. In the following graph, each vertex has been label with its EST, and the critical path is marked.



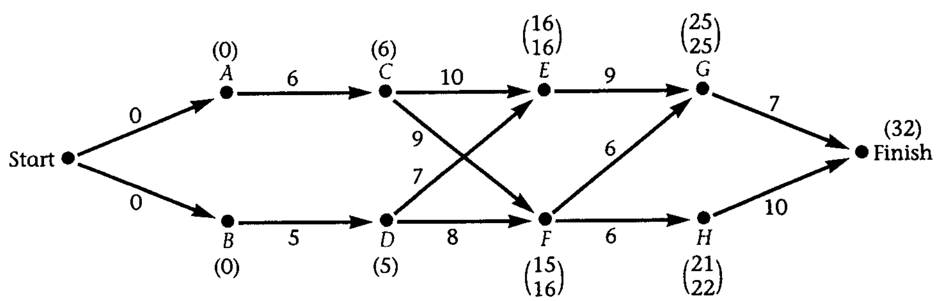
a. Task *E* can begin as early as day 9. If it begins on day 9, when will it be completed? If it begins on day 10? On day 11? What will happen if it begins on day 12?
b. What is the latest day on which task *E* can begin if task *G* is to begin on day 18?

If an activity is not on the critical path, it is possible for it to start later than its earliest-start time and not delay the project. The latest a task can begin without delaying the project's minimum completion time is known as the **latest-start time** (LST) for the task. For example, the LST for *E* is day 11.

c. In order to find the LST for vertex *C*, the times of the two vertices *D* and *E* need to be considered. Since vertex *D* is on the critical path, the latest it can start is day 10. For *D* to begin on time, what is the latest day on which *C* can begin? In part b, you found that the latest

E can start is day 11. In that case, what is the latest *C* can begin? From this information, what is the latest (LST) that *C* can begin without delaying *either* task *D* or *E*?

9. To find the LST for each task, it is necessary to begin with the Finish and work through the graph in reverse order to the Start. Each of the vertices in the following graph are labeled with their ESTs. The LSTs for several of the tasks have been calculated and are shown below the ESTs on the vertices. Find the LSTs for the remaining tasks.



10. Write an algorithm to find the LSTs for the tasks in a graph. Test your algorithm on the graph in Exercise 1.

Projects

11. Interview the yearbook sponsors in your school to find out how they organize the publication of your school's yearbook. Create a task table that shows the approximate times and prerequisite tasks that must be completed before your yearbook can go to the publisher. Design a graph with the EST for each task, and identify the critical path.

12. Use the Internet or other sources to research and report on businesses or people who use PERT or similar evaluation techniques for project planning.

Lesson 4.3

The Vocabulary and Representations of Graphs

Graphs have many applications in addition to critical path analysis. They are frequently used in social science, computer science, chemistry, biology, transportation, and communications. In the following lessons several of these applications are examined.

Recall that a graph is a set of points called vertices and a set of connecting lines called edges. Often graphs are used to model situations in which the vertices represent objects, and edges are drawn between the vertices on the basis of a particular relationship between the objects. Note that the important characteristics of a graph remain unchanged whether the edges are curved or straight.

Explore This

Case 1

Suppose the vertices of Figure 4.4 represent the starting five players on a high school basketball team, and the edges denote friendships. This graph

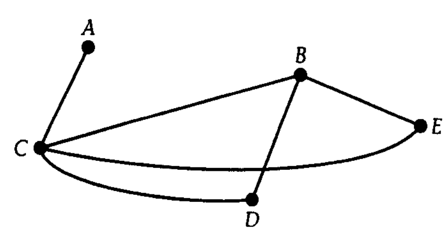


Figure 4.4 Graph showing five vertices and five edges.

indicates that player C is friends with all of the other players and that E has only two friends, C and B. Notice that edge CE and edge DB intersect in this graph, but the intersection does not create a new vertex.

1. Which player has only one friend?
2. How many friends does E have? Who are they?
3. Redraw the graph so that A has no friends.

Consider the following solutions.

1. Player A.
2. Two, C and B.
3. The graph shown in Figure 4.5.

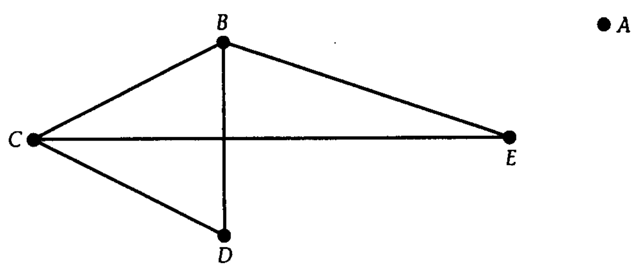
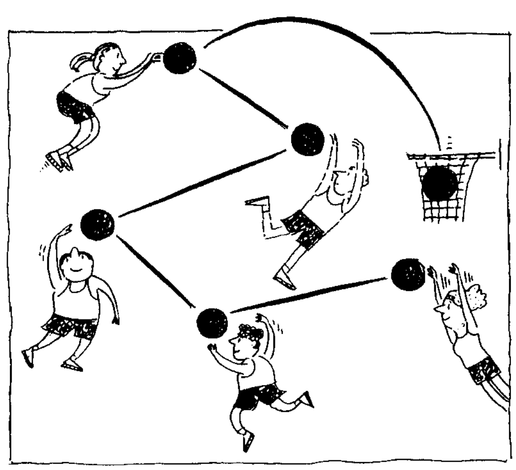


Figure 4.5 Graph that shows A with no friends.

A graph is **connected** if there is a path between each pair of vertices. The graph shown above in Figure 4.5 is not connected because there is no path from A to any of the other vertices.



Case 2

Figure 4.4 on page 164 could represent many different things other than basketball players and their friendships. For example, let the vertices in the figure represent rooms in your school. The vertices are connected if there are direct hallways between two rooms. Then, according to the graph in Figure 4.4, a student can get from room *C* directly to any of the other four rooms.

When two vertices are connected with an edge, they are said to be **adjacent**. In Figure 4.4, *C* is adjacent to *A*, *B*, *D*, and *E*. Although there is no single edge from *D* to *A*, it is possible to get from room *D* to room *A* by following a path that goes through room *C*. Although a path exists between *D* and *A*, they are not adjacent.

Try drawing a graph in which there is direct access from each room to every other room. Figure 4.6 shows two possible ways to represent this graph. Even though these graphs appear to be different, they are structurally the same, and so they are considered the same graph. It is important to note that there is no single correct way to draw a graph to represent a given situation. A good graph is one that clearly represents the information needed to solve some problem.

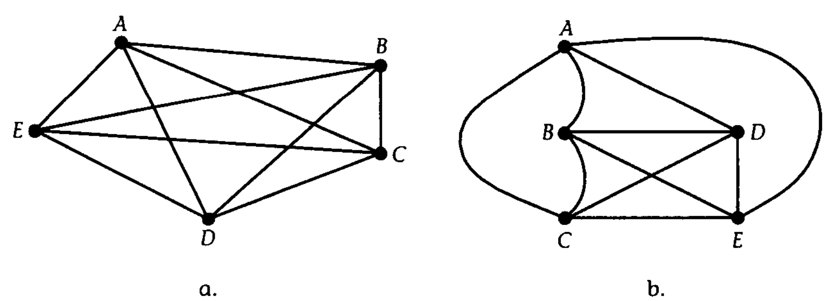


Figure 4.6 Two different representations of five rooms in a school, with each having direct access to every other room.

Graphs such as the ones in Figure 4.6, in which every pair of vertices is adjacent, are called **complete** graphs. Complete graphs are often denoted by K_n, where n is the number of vertices in the graph. Figure 4.6 depicts a K_5 graph.

Alternative Representations

Graphs can be represented in several different ways. A diagram, such as the one in Figure 4.4, is just one of these ways. Another way to represent the information is to list the set of vertices and the set of edges. For example, the graph in Figure 4.4 on page 164 can be described by

Vertices = {A, B, C, D, E} Edges = {AC, CB, CE, CD, BD, BE}.

A third way to represent this information is with an adjacency matrix. This type of representation can be used to depict the vertices and edges of the graph in a computer or calculator.

A 5 × 5 matrix in which both the rows and columns correspond to the vertices *A*, *B*, *C*, *D*, and *E* can be used to represent Figure 4.4. If an edge exists between vertices, a 1 appears in the corresponding position in the matrix; otherwise a 0 appears.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The entry in row 2, column 4 is a 1. This indicates that vertices *B* and *D* are adjacent; that is, an edge exists between them.

Exercises

1. Mr. Butler bought six different types of fish. Some of the fish can live in the same aquarium, but others cannot. Guppies can live with Mollies, Swordtails can live with Guppies, Plecostomi can live with both Mollies and Guppies, Gold Rams can live only with Plecostomi, and Piranhas cannot live with any of the other fish. Draw a graph to illustrate this.
2. Construct a graph for each of the following sets of vertices and edges. Which of the graphs are connected? Which are complete?

   a. V = {A, B, C, D, E}
   E = (AB, AC, AD, AE, BE}.

   b. V = {M, N, O, P, Q, R, S}
   E = {MN, SR, QS, SP, OP}.

   c. V = {E, F, G, J, K, M}
   E = {EF, KM, FG, JM, EG, KJ}.

   d. V = {W, X, Y, Z}
   E = {WX, XZ, YZ, XY, WZ, WY}.

3. Draw a graph with vertices = {*A*, *B*, *C*, *D*, *E*, *F*} and edges = {*AB*, *CD*, *DE*, *EC*, *EF*}.

a. Name two vertices that are not adjacent.

b. *F*, *E*, *C* is one possible path from *F* to *C*. This path has a length of 2, since two edges were traveled to get from *F* to *C*. Name a path from *F* to *C* with a length of 3.

c. Is this graph connected? Explain why or why not?

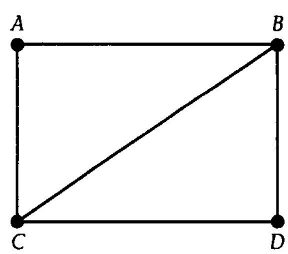
d. Is this graph complete? Explain why or why not?

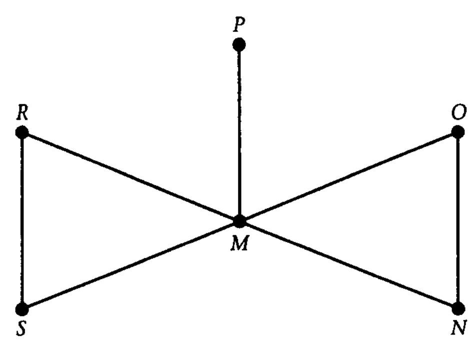
4. Draw a graph with five vertices in which vertex *W* is adjacent to *Y*, *X* is adjacent to *Z*, and *V* is adjacent to each of the other vertices.

5. Construct a graph for each adjacency matrix. Label the vertices *A*, *B*, *C*, ...

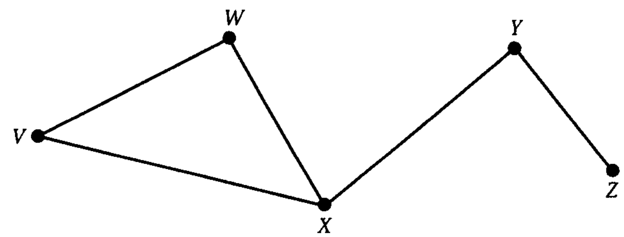
a. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

6. Create an adjacency matrix for each of the following graphs:

a.


b.


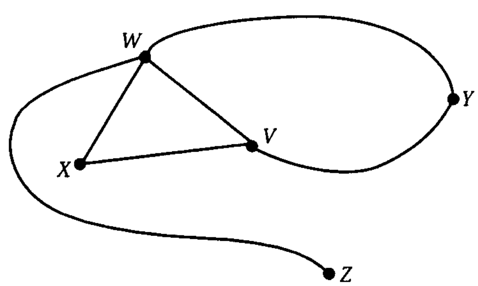
7. Give the adjacency matrix for the following graph.



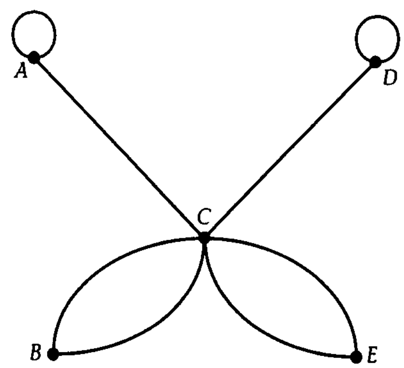
a. What do you notice about the main diagonal of the matrix?
b. Does your matrix possess symmetry? If so, where?
c. If an adjacency matrix has a 1 on the main diagonal, what would that indicate? What would a 2 in row 2, column 1 indicate?

8. Find the sum of each row of your matrix from Exercise 7. What do these sums tell you about the graph of the matrix?

9. In a graph, the number of edges that have a specific vertex as an endpoint is known as the **degree** or **valence** of that vertex. In the following graph, the degree of vertex W is 4. This is denoted by $\deg(W) = 4$. Find the degrees of each of the other vertices.



10. An edge that connects a vertex to itself is called a **loop**. If a graph contains a loop or **multiple edges** (more than one edge between two vertices), the graph is known as a **multigraph**. When finding the degree of a vertex on which there is a loop, the loop is counted twice. For example, deg(A) = 3.



a. Find the degree of vertices B, C, D, and E.
b. Give the adjacency matrix for the above multigraph.
c. Compare an adjacency matrix for a graph and one for a multigraph. Without seeing the graph, can you tell which belongs to the graph and which belongs to the multigraph? Explain how you know.

11. Complete the following table for the sum of the degrees of the vertices in a complete graph.

Graph	Number of Vertices	Sum of the Degrees of All of the Vertices	Recurrence Relation
K_1	1	0	
K_2	2	2	$T_2 = T_1 + 2$
K_3	3	6	$T_3 = T_2 + 4$
K_4	—	—	—
K_5	—	—	—
K_6	—	—	—

Write a recurrence relation that expresses the relationship between the sum of the degrees of all the vertices for K_n and the sum for K_{n-1}.

12. Having completed the table in Exercise 11, what did you notice about the sum of the degrees of the vertices for any complete graph? Do you think this is true for any graph? If so, explain why this is true; if not, give a counterexample.

13. a. Try to construct a graph with four vertices, two of the vertices with degree 3 and two with degree 2. No loops or multiple edges may be used.

b. Try to construct a graph with five vertices, three of the vertices with an odd degree and two with even degree. No loops or multiple edges may be used.

c. What do you think might be true about the number of vertices with even degree and the number of vertices with odd degree in any graph? (Hint: Try a few examples to check your hypothesis.)

14. Describe the adjacency matrix of a complete graph.

15. Complete the following table for the given complete graphs.

Graph	Number of Vertices	Number of Edges	Recurrence Relation
K_1	1	0	
K_2	2	1	$S_2 = S_1 + 1$
K_3	3	3	$S_3 = S_2 + 2$
K_4	—	—	—
K_5	—	—	—
K_6	—	—	—

Write a recurrence relation that expresses the relationship between the number of edges of K_n and the number of edges of K_{n-1}.

16. Central High School is a member of a five-team hockey league. Each team in the league plays exactly two games, which must be against different teams. Show that there is only one possible graph for this schedule.

Lesson 4.4

Euler Circuits and Paths

Now that you are familiar with some of the concepts of graphs and the way graphs convey connections and relationships, it's time to begin exploring how they can be used to model many different types of situations.

Explore This

Consider the graph in Figure 4.7. Try to draw this figure without lifting your pencil from the paper and without tracing any of the lines more than once. Is this possible?

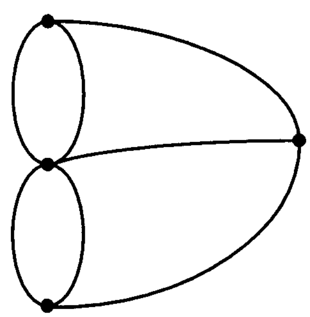


Figure 4.7 Graph.

The graph in Figure 4.7 represents an eighteenth-century problem that intrigued the famous Swiss mathematician Leonhard Euler (pronounced "oiler"). The problem was one that had been posed by the residents of Königsberg, a city in what was then Prussia but is now the Russian city of Kaliningrad. In the 1700s, seven bridges connected two islands in the Pregel River to the rest of the city (see Figure 4.8). The people of Königsberg wondered whether it would be possible to walk through the city by crossing each bridge exactly once and return to the original starting point.

Mathematician of Note

Leonard Euler (1707–1783). Euler was an extraordinary mathematician who published over 500 works during his lifetime. Even total blindness for the last 17 years of his life did not stop his effectiveness and genius. He is often referred to as "the father of graph theory."

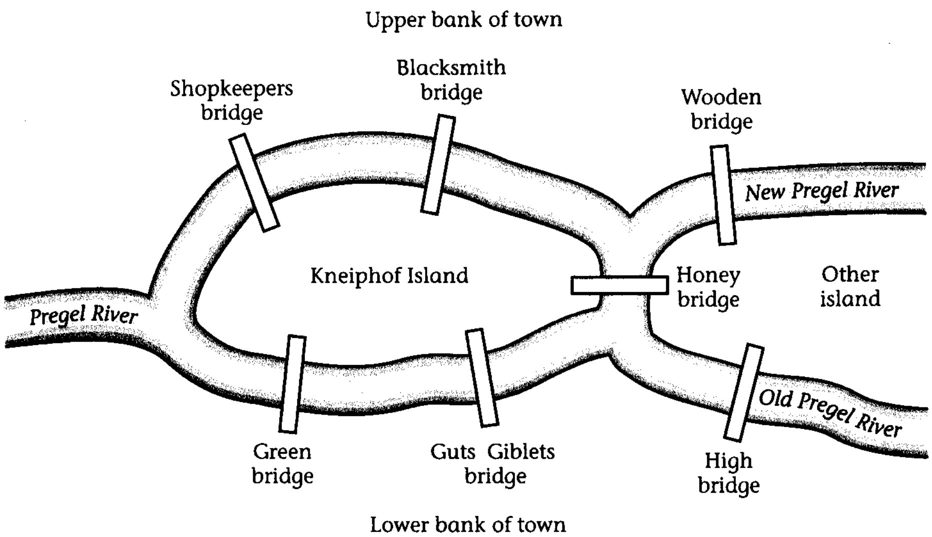


Figure 4.8 Representation of the seven bridges of Königsberg.

Using a graph like the one in Figure 4.7, in which the vertices represented the landmasses of the city and the edges represented the bridges, Euler found that it was not possible to make the desired walk through the city. In so doing, he also discovered a solution to problems of this general type.

What did Euler find? Try to reproduce the graphs in Figure 4.9 without lifting your pencil or tracing the lines more than once.

1. When can you draw the figures without retracing any edges and still end up at your starting point?
2. When can you draw the figure without retracing and end up at a point other than the one from which you began?
3. When can you not draw the figure without retracing?

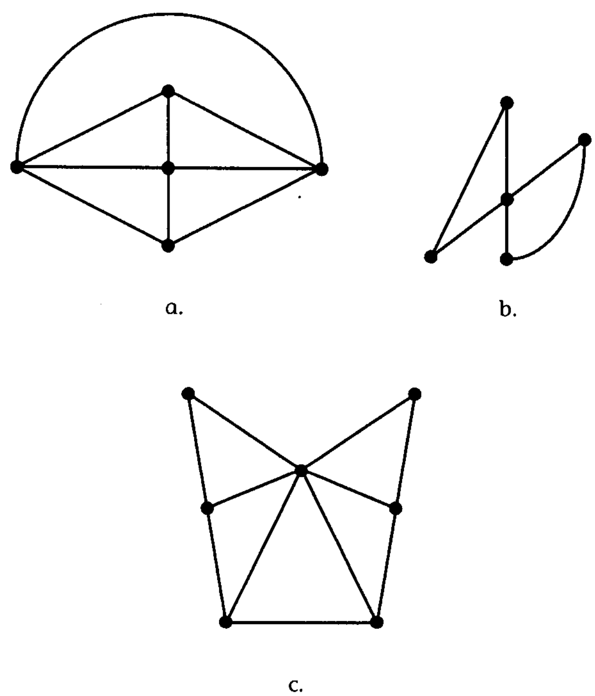


Figure 4.9 Graphs to trace.

Euler found that the key to the solution was related to the degrees of the vertices. Recalling that the degree of a vertex of a graph is the number of edges that have that vertex as an endpoint, find the degree of each vertex of the graphs in Figure 4.9. Do you see what Euler noticed?

Euler hypothesized and later proved that in order to be able to traverse each edge of a connected graph exactly once and to end at the starting vertex, the degree of each vertex of the graph must be even, as in Figure 4.9b. In honor of Leonhard Euler, a path that uses each edge of a graph exactly once and ends at the starting vertex is called an **Euler circuit**.

Euler also noticed that if a connected graph had exactly two odd vertices, it was possible to use each edge of the graph exactly once but to end at a vertex different from the starting vertex. Such a path is called an **Euler path**. Figure 4.9a is an example of a graph that has an Euler path. Figure 4.9c has four odd vertices, and so it cannot be traced without lifting your pencil. It has neither an Euler circuit nor an Euler path.

An Euler circuit for a relatively small graph can usually be found by trial and error. However, as the number of vertices and edges increases, a systematic way of finding the circuit becomes necessary. The following algorithm gives a procedure for finding an Euler circuit for a connected graph with all vertices of even degree.

Euler Circuit Algorithm

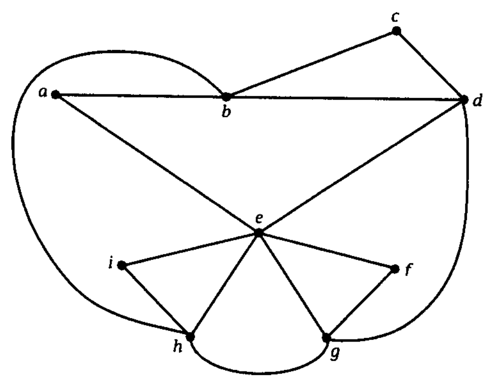
1. Pick any vertex, and label it *S*.
2. Construct a circuit, *C*, that begins and ends at *S*.
3. If *C* is a circuit that includes all edges of the graph, go to step 8.
4. Choose a vertex, *V*, that is in *C* and has an edge that is not in *C*.
5. Construct a circuit *C′* that starts and ends at *V* using edges not in *C*.
6. Combine *C* and *C′* to form a new circuit. Call this new circuit *C*.
7. Go to step 3.
8. Stop. *C* is an Euler circuit for the graph.

Euler Circuit Algorithm

1. Pick any vertex, and label it S.
2. Construct a circuit, C, that begins and ends at S.
3. If C is a circuit that includes all edges of the graph, go to step 8.
4. Choose a vertex, V, that is in C and has an edge that is not in C.
5. Construct a circuit C' that starts and ends at V using edges not in C.
6. Combine C and C' to form a new circuit. Call this new circuit C.
7. Go to step 3.
8. Stop. C is an Euler circuit for the graph.

Example

Use the Euler circuit algorithm to find an Euler circuit for the following graph.



- Apply step 1 of the algorithm. Choose vertex b, and label it S.
- Let C be the circuit S, c, d, e, a, S.
- Circuit C does not contain all edges of the graph, so proceed to step 4 of the algorithm.
- Choose vertex d.
- Let C' be the circuit d, g, h, S, d.
- Combine C and C' by replacing vertex d in the circuit C with the circuit C'. Let C now be the circuit $S, c, d, g, h, S, d, e, a, S$.
- Go to step 3 of the algorithm.
- Circuit C does not contain all edges of the graph, so again proceed to step 4.
- Choose vertex g.
- Let C' be the circuit g, f, e, i, h, e, g
- Combine C and C' by replacing vertex g in the circuit C with the circuit C'. Let C now be the circuit $S, c, d, g, f, e, i, h, e, g, h, S, d, e, a, S$.

- Circuit C now contains all edges of the graph, so go to step 8 of the algorithm and stop. C is an Euler circuit for the graph.

Edges with Direction

Many applications of graphs require that the edges have direction. A city with one-way streets is one such example. A graph that has directed edges, edges that can be traversed in only one direction, is known as a **digraph** (see Figure 4.10). The number of edges coming into a vertex is known as the **indegree** of the vertex, and the number of edges going out of a vertex is known as the **outdegree**.

Examine Figure 4.10. This digraph can be described by

Vertices = {A, B, C, D} Ordered edges = {AB, BA, BC, CA, DB, AD}.

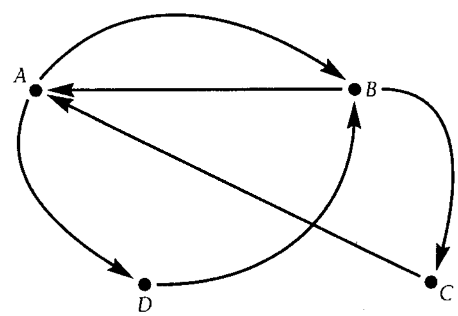


Figure 4.10 Digraph.

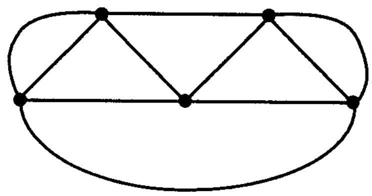
If you follow the indicated direction of each edge, is it possible to start at some vertex, draw the digraph, and end up at the vertex from which you started? That is, does this digraph have a directed Euler circuit?

Check the indegree and outdegree of each vertex. You will find that a connected digraph has an Euler circuit if the indegree and outdegree of each vertex are equal.

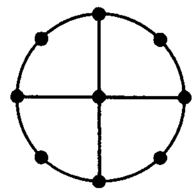
Exercises

1. State whether each graph has an Euler circuit, an Euler path, or neither. Explain why.

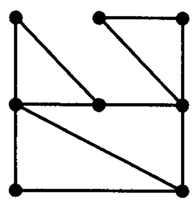
a.



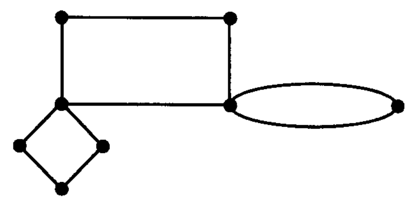
b.



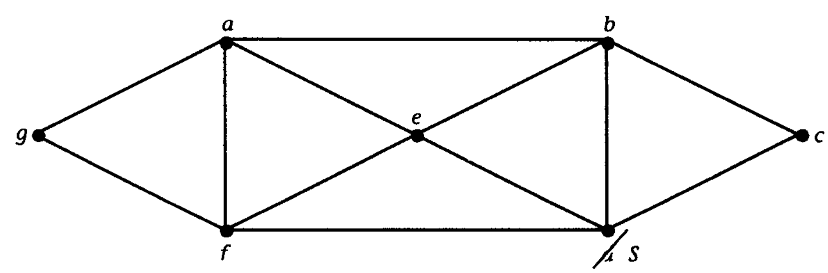
c.



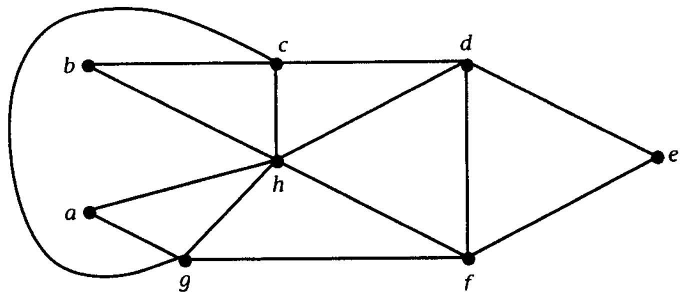
d.



2. Sally began using the Euler circuit algorithm to find the Euler circuit for the following graph. She started at vertex *d* and labeled it *S*. The first circuit she found was *S*, *e*, *f*, *a*, *b*, *c*, *S*. Using Sally's start, continue the algorithm and find an Euler circuit for the graph.



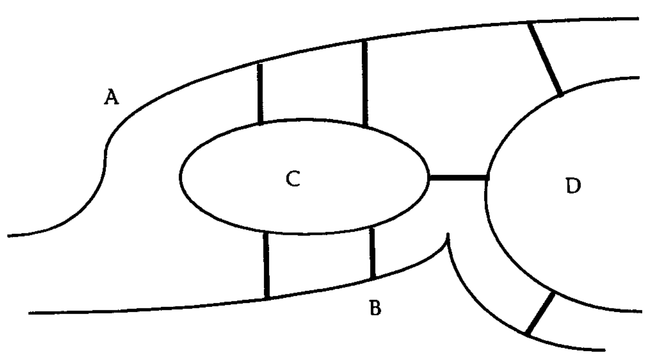
3. Use the Euler circuit algorithm to find an Euler circuit for the following graph.



4. The text states that to apply the Euler circuit algorithm, the graph must be "connected with all vertices of even degree."
a. Why is it necessary to state that the graph must be connected?
b. Give an example of a graph with all vertices of even degree that does not have an Euler circuit.

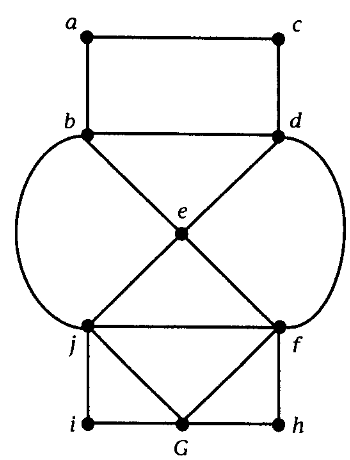
5. Will a complete graph with 2 vertices have an Euler circuit? With 3 vertices? With 4 vertices? With 5 vertices? With *n* vertices?

6. The present-day Königsberg has two more bridges than it did in Euler's time. One bridge was added to connect the two banks on the river, *A* to *B* in the following figure. Another one was added to link the land to one of the islands, *B* to *D*. Is it now possible to make the famous walk and return to the starting point? Explain your reasoning.



Königsberg's *original* seven bridges.

7. The street network of a city can be modeled with a graph in which the vertices represent the street corners, and the edges represent the streets. Suppose you are the city street inspector and it is desirable to minimize time and cost by not inspecting the same street more than once.



a. In this graph of the city, is it possible to begin at the garage (G) and inspect each street only once? Will you be back at the garage at the end of the inspection?

b. Find a route that inspects all streets, repeats the least number of edges possible, and returns to the garage.

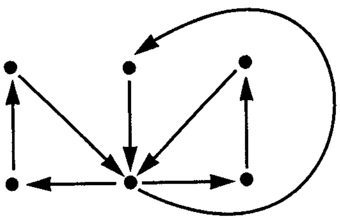
8. Construct the following digraphs.

a. $V = \{A, B, C, D, E\}$
$E = \{AB, CB, CE, DE, DA\}$.

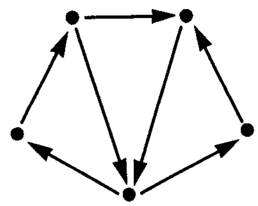
b. $V = \{W, X, Y, Z\}$
$E = \{WX, XZ, ZY, YW, XY, YX\}$.

9. Determine whether the digraph has a directed Euler circuit.

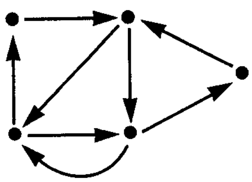
a.



b.



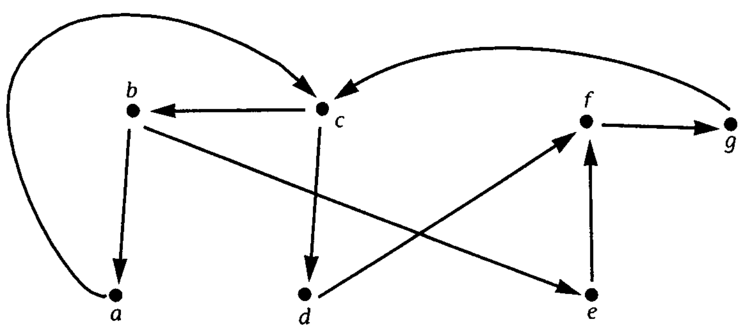
c.



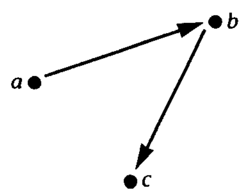
10. a. Does the following digraph have a directed Euler circuit? Explain why or why not.

b. Does it have a directed Euler path? If it does, which vertices can be the starting vertex?

c. Write a general statement explaining when a digraph has a directed Euler path.



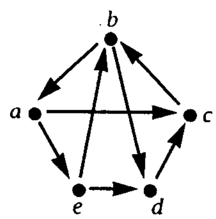
11. A digraph can be represented by an adjacency matrix. If there is a directed edge from vertex a to vertex b, then a 1 is placed in row a, column b of the matrix; otherwise a 0 is entered. Matrix M is the adjacency matrix for the following graph.



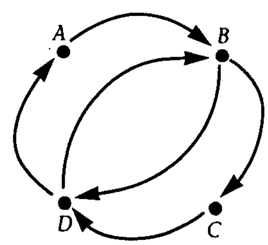
$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the adjacency matrix for each of the following digraphs.

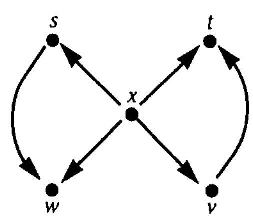
a.



b.



c.



12. a. Construct a digraph for the following adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b. Is there symmetry along the main diagonal of the adjacency matrix? Explain why or why not.

c. Find the sum of the numbers in the second row. What does that total indicate?

d. Find the sum of the numbers in the second column. What does that total indicate?

Computer/Calculator Explorations

13. Create a computer or calculator program that prompts the user to enter the adjacency matrix for a connected graph. The program should then tell the user whether or not the graph has an Euler circuit.

Projects

14. Leonhard Euler was known for many accomplishments in addition to his discoveries related to graph theory. After researching Euler's achievements, create a "biographic poster" that illustrates the important milestones of his life.

15. Research and report on algorithms that determine Euler circuits for graphs that have them.

Lesson 4.5

Hamiltonian Circuits and Paths

Since its inception, graph theory has been closely tied to applications. In Lesson 4.4, you investigated situations in which you needed to traverse each edge of a graph. In this lesson, you will explore reasons to examine the vertices.

Explore This

Suppose once again that you are a city inspector, but instead of inspecting all of the streets in an efficient manner, you must inspect the fire hydrants that are located at each of the street intersections. This implies that you are searching for an optimal route that begins at the garage *G*, visits each intersection exactly once, and returns to the garage (see Figure 4.11).

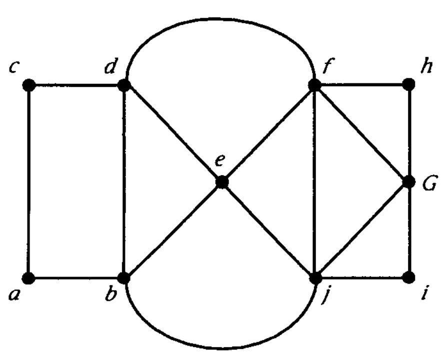
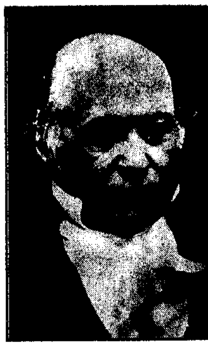


Figure 4.11 Street network.

One path that meets these criteria is *G*, *h*, *f*, *d*, *c*, *a*, *b*, *e*, *j*, *i*, *G*. Notice that it is not necessary to traverse every edge of the graph when visiting each vertex exactly once.

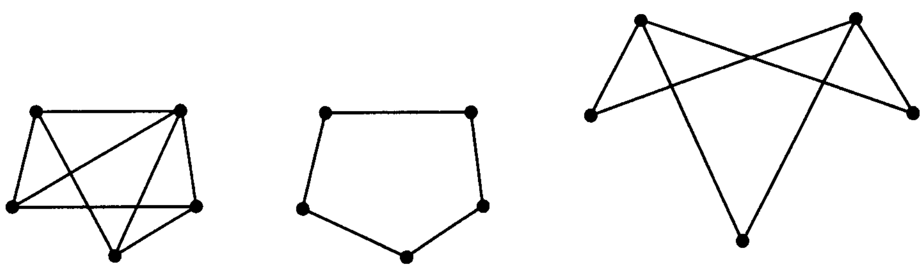
In 1856, Sir William Rowan Hamilton used his mathematical knowledge to create a game called the Icosian game. The game consisted of a graph in which the vertices represented major cities in Europe, and the object of the game was to find a path that visited each of the 20 vertices exactly once. In honor of Hamilton and his game, a path that uses each vertex of a graph exactly once is known as a **Hamiltonian path**. If the path ends at the starting vertex, it is called a **Hamiltonian circuit**.

Try to find a Hamiltonian circuit for each of the graphs in Figure 4.12.



Mathematician of Note

Sir William Rowan Hamilton (1805–1865). Hamilton, a leading nineteenth-century Irish mathematician, was appointed Astronomer Royal of Ireland at the age of 22 and knighted at 30. His most notable discoveries were in algebra.



a. b. c.

Figure 4.12 Graphs with possible Hamiltonian circuits.

Mathematicians continue to be intrigued with this type of problem because a simple test for determining whether a graph has a Hamiltonian circuit has not been found, and it now appears that a general solution that applies to all graphs may be impossible. Fortunately, several theorems have been proved that help guarantee the existence of Hamiltonian circuits for certain kinds of graphs. The following is one such theorem.

If a connected graph has n vertices, where $n > 2$ and each vertex has degree of a least $n/2$, then the graph has a Hamiltonian circuit.

To apply the Hamiltonian circuit theorem to Figure 4.12a on page 185, check the degree of each vertex. Since each of the five vertices of the graph has degree of at least 5/2, the theorem guarantees that the graph has a Hamiltonian circuit. Unfortunately, it does not tell you how to find it.

The theorem has another downside as well. If a graph has a vertex with degree less than $n/2$, then the theorem simply does not apply to that graph. It may or it may not have a Hamiltonian circuit. Each of the graphs in parts b and c of Figure 4.12 has some vertices of degree less than 5/2, so no conclusions can be drawn. By inspection, Figure 4.12b has a Hamiltonian circuit, but Figure 4.12c does not.

Tournaments

As with Euler circuits, it often is useful for the edges of the graph to have direction. Consider a competition in which each player must play every other player. By using directed edges, it is possible to indicate winners and losers. To illustrate this, draw a complete graph in which the vertices represent the players, and a directed edge from vertex *A* to vertex *B* indicates that player *A* defeats player *B*. This type of graph is known as a tournament. A **tournament** is a digraph that results from giving directions to the edges of a complete graph. Figure 4.13 shows a tournament in which *A* beats *B*, *C* beats *B*, and *A* beats *C*.

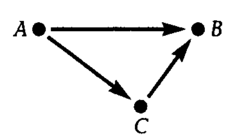


Figure 4.13 Tournament with three vertices.

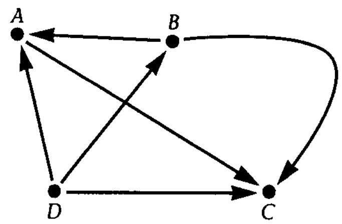
One interesting property of such a digraph is that every tournament contains at least one Hamiltonian path. If there is exactly one such path, it can be used to rank the teams in order, from winner to loser.

Example

Suppose four teams play in the school soccer round-robin tournament. The results of the competition follow:

Game	AB	AC	AD	BC	BD	CD
Winner	B	A	D	B	D	D

Draw a digraph to represent the tournament. Find a Hamiltonian path and use it to rank the participants from winner to loser.

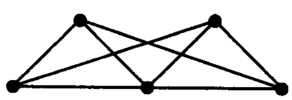


To find a solution, remember that a tournament results from a *complete* graph when direction is given to the edges. In this case, there is only one Hamiltonian path for the graph: *D*, *B*, *A*, *C*. Therefore, *D* finishes first, *B* is second, *A* is third, and *D* finishes fourth.

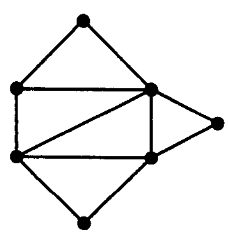
Exercises

1. Apply the theorem from page 185 to the graphs below. According to the theorem, which of the graphs have Hamiltonian circuits? Explain your reasoning.

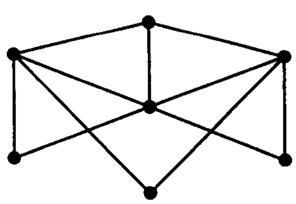
a.



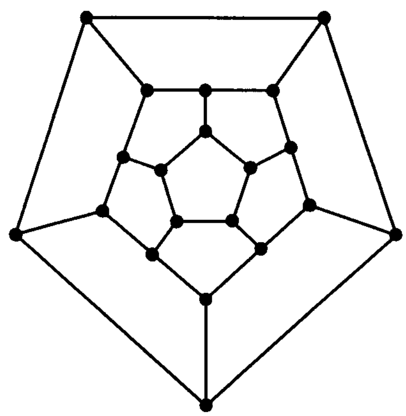
b.



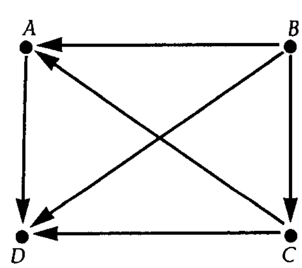
c.

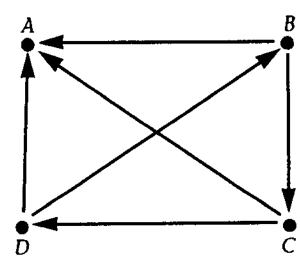


2. Give two examples of situations that could be modeled by a graph in which finding a Hamiltonian path or circuit would be of benefit.
3. a. Construct a graph that has both an Euler and a Hamiltonian circuit.
   b. Construct a graph that has neither an Euler nor a Hamiltonian circuit.
4. Hamilton's Icosian game was played on a wooden regular dodecahedron (a solid figure with 12 sides). Here is a planar representation of the game.



   a. Copy the graph onto your paper and find a Hamiltonian circuit for the graph.
   b. Is there only one Hamiltonian circuit for the graph?
   c. Can the circuit begin at any of the vertices or only some of them?
5. Draw a tournament with five players, in which player *A* defeats everyone, *B* defeats everyone but *A*, *C* is defeated by everyone, and *D* defeats *E*.
6. Find all the directed Hamiltonian paths for each of the following tournaments:

a. 

b. 

7. Draw a tournament with three vertices in which:
   a. One player wins all the games he or she plays.
   b. Each player wins exactly one game.
   c. Two players lose all of the games they play.

8. Draw a tournament with five vertices in which there is a three-way tie for first place.

9. When ties exist in a ranking for a tournament (e.g., more than one first place winner), there is more than one Hamiltonian path for the graph. Explain why this is so.

10. a. Write an algorithm that uses the outdegree of the vertices to find the Hamiltonian path for a tournament that has exactly one Hamiltonian path.
   b. Explain the difficulties that arise with your algorithm when the tournament has more than one Hamiltonian path.

11. Complete the following table for a tournament.

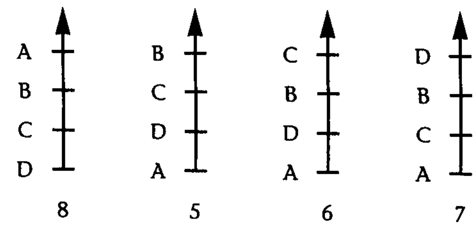
Number of Vertices	Sum of the Outdegrees of the Vertices
1	0
2	1
3	3
4	
5	
6	

Write a recurrence relation that expresses the relationships between S_n, the sum of the outdegrees for a tournament with n vertices, and S_{n-1}.

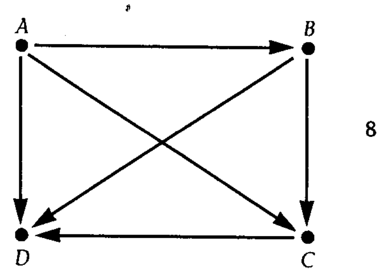
12. In a tournament a **transmitter** is a vertex with a positive outdegree and a zero indegree. A **receiver** is a vertex with a positive indegree and a zero outdegree. Explain why a tournament can have at most one transmitter and at most one receiver.

13. Use mathematical induction on the number of vertices to prove that every tournament has a Hamiltonian path.
   a. Begin the mathematical induction process by showing that every tournament with one vertex has a Hamiltonian path.
   b. Assume that a tournament of k vertices has a Hamiltonian path and use this assumption to prove that a tournament of $k + 1$ vertices has a Hamiltonian path.

14. Consider the set of preference schedules from Lesson 1.3:



The first preference schedule could be represented by the following tournament.



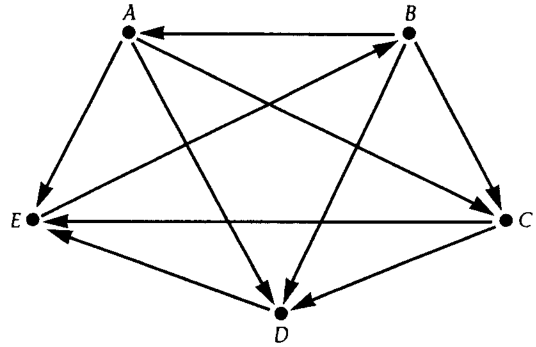
a. Construct tournaments for each of the three other preference schedules.

b. Construct a cumulative preference tournament that would show the overall results of the four individual preference schedules.

c. Is there a Condorcet winner in the election? (Recall from Lesson 1.3 that a Condorcet winner is one who is able to defeat each of the other choices in a one-on-one contest.)

d. Find a Hamiltonian path for the cumulative tournament. What does this path indicate?

15. a. Construct an adjacency matrix for the following digraph, and call the matrix *M*.



By summing the rows of M, you can see that a tie exists between A and B, each with three wins.

b. Square M. Notice that this new matrix M^2 gives the number of paths of length 2 between vertices. For example, the 3 in entry M_{25} indicates that 3 paths of length 2 exist between B and E. These paths are B, C, E; B, D, E; and B, A, E. In the case of a tournament, this means that B has three 2-stage wins, or **dominances,** over E.

c. Add M and M^2. Use the sum to determine the total number of ways that A and B can dominate in one and two stages. Who might now be considered the winner?

d. What would M^3 indicate? Find M^3 and see whether you are correct.

Projects

16. Design and build a Planar Icosian Game by enlarging the graph in Exercise 4 and copying it onto a piece of plywood or heavy cardboard. Use tacks for the vertices. The game is then played by tying a piece of string (approximately 12 inches or longer) on one of the tacks (vertices) and attempting to wind the string from tack to tack (following the lines on the graph) until all of the tacks are touched and the player is back to the initial tack (in other words, until a Hamiltonian circuit is found). Try the game with younger children and adults. Who seems to find a Hamiltonian circuit the quickest?

Lesson 4.6

Graph Coloring

When it comes time to schedule meetings at school or register students into classes, problems often arise. Mathematicians have found that graphs are useful tools in helping to resolve conflicts of these types.

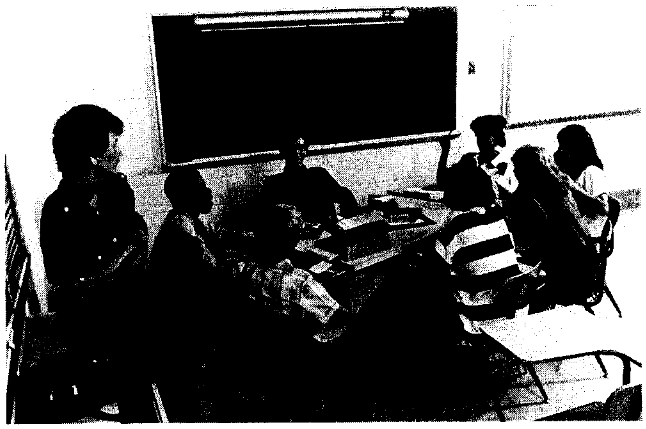
Explore This

Here is a table of clubs at Central High School and students who hold offices in these clubs.

	Math Club	Honor Club	Science Club	Art Club	Pep Club	Spanish Club
Matt	X	X	X	—	—	—
Marty	X	—	—	X	X	—
Kim	—	X	—	—	—	X
Lois	X	—	X	—	—	—
Dot	X	—	—	—	X	—

Each club at Central High wants to meet once a week. Since several students hold offices in more than one organization, it is necessary to arrange the meeting days so that no students are scheduled for more than one meeting on the same day. Is it possible to create such a schedule? What is the minimum number of days needed?

One possible solution to the problem is to use five days for the scheduling. The Math and Spanish Clubs could meet on Monday, and the remaining clubs could meet on the other four days. If the problem is to schedule the meetings in the fewest number of days, then the solution of using five days is not optimal. It is possible to create a schedule using only three days.



Graphs can be of help when scheduling high school activities.

Finding such a schedule by trial and error is not difficult in this case, but a mathematical model would be helpful for more complicated problems. The first step in creating such a model might be to construct a graph in which the vertices represent clubs at Central High and the edges indicate conflict. If an edge joins two vertices, then those two clubs share an officer and cannot meet on the same night.

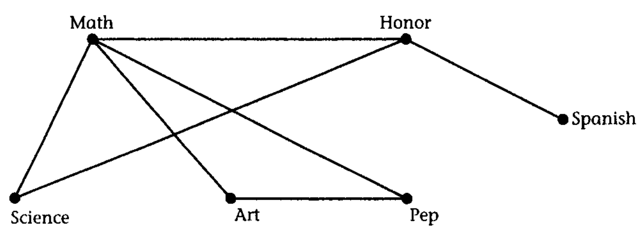


Figure 4.14 Graph showing the organizations of Central High that share an officer.

After completing the graph (see Figure 4.14), label the vertices with days of the week. When doing so, keep in mind that adjacent vertices must have different labels, since this is where the conflicts occur. One way of assigning days is to begin with the Math Club and label it with Monday. Look at the vertices not adjacent to the Math Club vertex. Since no one belongs to both the Math Club and the Spanish Club, also label the Spanish Club with Monday. Label the Honor Club with Tuesday. The Pep or Art

Club, but not both, can receive a Tuesday label. The one not labeled Tuesday is then labeled Wednesday, the third day. The resulting schedule is an optimal solution to the problem, but notice that it is not unique.

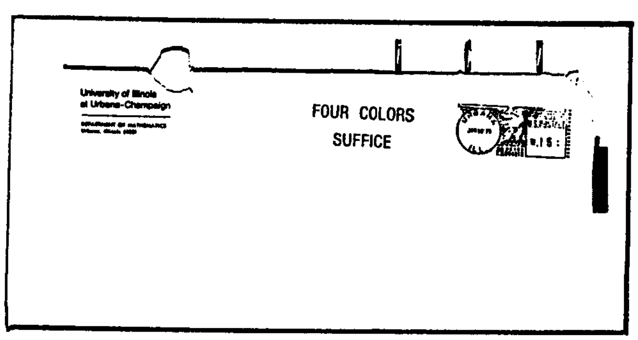
Problems of this type are called *coloring* problems because historically the labels placed on the vertices of the graphs were referred to as *colors*. The process of labeling the graph is called **coloring the graph**, and the minimum number of labels, or colors, that can be used is known as the **chromatic number** of the graph. The chromatic number for the graph in Figure 4.14 is 3.

Questions of this type first attracted interest in the nineteenth century when mathematicians such as Augustus de Morgan, William Rowan Hamilton, and Arthur Cayley became interested in a problem known as the four-color conjecture. The problem stated that any map that could be drawn on the surface of a sphere could be colored with, at most, four colors.

For over 100 years, this problem intrigued mathematicians. During that time, many claimed to have proved the conjecture, but flaws were always found in the proofs. It wasn't until 1976 that Kenneth Appel and Wolfgang Haken of the University of Illinois solved the famous problem, and the four-color conjecture became known as the four-color theorem.

The problem was proved in a way very different from earlier attempts. Appel and Haken used a high-speed computer in their verification. When the proof was finally complete, they had used over 1,200 hours of computer time and had examined approximately 1,936 basic forms of maps. Because of the unusual method of proof and the inability to emulate it by hand, criticism arose from many in the mathematical community. A recent simplification of the four-color theorem proof, by Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas, has removed the cloud of doubt hanging over the complex original proof of Appel and Haken.

Are you wondering what scheduling classes and coloring maps have in common? As it turns out, they are both problems that can be solved with the help of graph coloring techniques. One way to approach the problem of coloring a map is to represent each region of the map with a

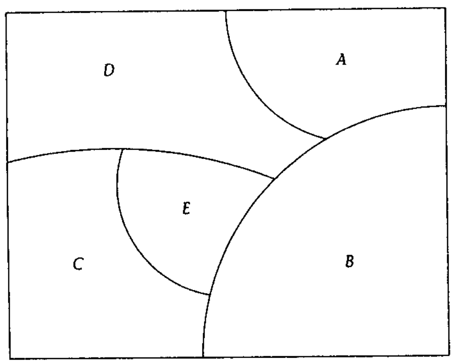


Postage meter stamp used by the University of Illinois to commemorate the proof of the four-color theorem.

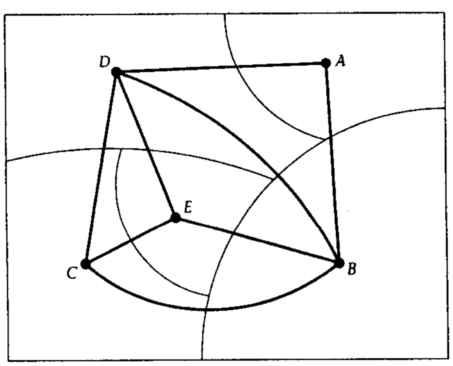
vertex of a graph. Two vertices are then connected by an edge if the regions they represent have a common border. The process of coloring the graph resulting from a map is the same as when you colored the scheduling graph model.

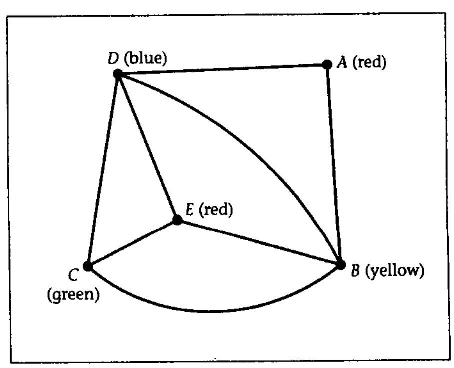
Example

Color the following map using four or fewer colors.



To find a solution, represent the map with a graph in which each vertex represents a region of the map, and draw edges between vertices if the regions on the map have a common border. Then label the graph using a minimum number of colors.



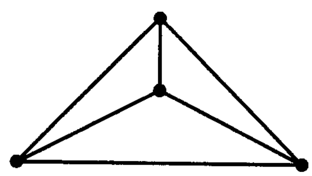


Four colors are necessary to color this map.

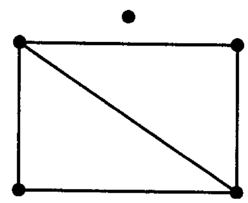
Exercises

1. Find the chromatic number for each of the following graphs.

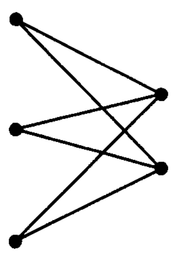
a.



b.



c.

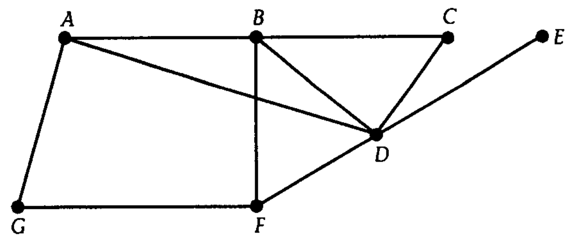


2. a. Draw a graph that has four vertices and a chromatic number of 3.
b. Draw a graph that has four vertices and a chromatic number of 1.

3. As the number of vertices in a graph increases, a systematic method of labeling (coloring) the vertices becomes necessary. One way to do this is to create a coloring algorithm.

a. It is possible to begin the coloring process in several different ways, but one way is to color first the vertices with the most conflict. How can the vertices be ranked from those with the most conflict to those with the least?
b. After having colored the vertex with the most conflict, which other vertices can receive that same color?
c. Which vertex would then get the second color? Which other vertices could get that same second color?
d. When would the coloring process be complete?
e. Refer back to parts a to d of this exercise and create an algorithm that colors a graph.

4. Use the algorithm that you developed in Exercise 3 to color the following graph. What is the chromatic number of the graph? Did your algorithm find the correct chromatic number?

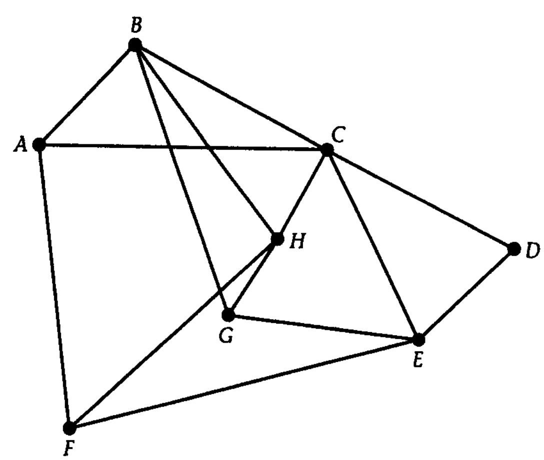


5. a. What is the chromatic number of K_2? K_3? K_4? K_5?
b. What can you say about the number of colors needed to color a complete graph? Explain your reasoning.
c. Draw a complete graph with four vertices. Use your algorithm from Exercise 3 to color it. Does your algorithm give the correct chromatic number?

6. A **cycle** is a path that begins and ends at the same vertex and does not use any edge or vertex more than once.
a. If a cycle has an even number of vertices, what is its chromatic number?
b. What is the chromatic number of a cycle with an odd number of vertices?
c. Draw a cycle with six vertices. Use your algorithm to color it. Does your algorithm give the correct chromatic number?

7. If your algorithm failed to give a minimal coloring for one of the graphs you tried, do not be too concerned, because it does not necessarily mean that you have a poor algorithm. Mathematicians continue to search for "good" coloring algorithms, but so far, they have been unable to find one that colors every graph in the fewest number of colors possible. If you've not found a graph that causes your algorithm to fail, try to draw a graph that will do so.

8. Mrs. Suzuki is planning to take her history class to the art museum. Following is a graph showing those students who are not compatible. Assuming that the seating capacity of the cars is not a problem, what is the minimum number of cars necessary to take the students to the museum?



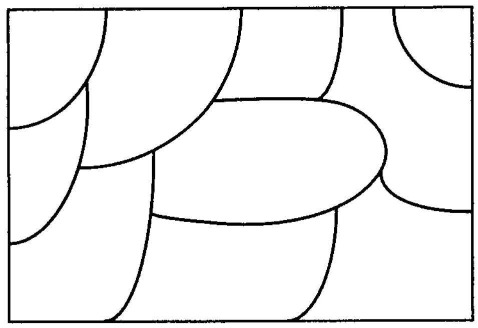
9. Refer to Exercise 1, Lesson 4.3, page 167. What is the minimum number of fish tanks needed to house the fish?

10. Following is a list of chemicals and the chemicals with which each cannot be stored.

Chemicals	Cannot Be Stored with
1	2, 5, 7
2	1, 3, 5
3	2, 4
4	3, 7
5	1, 2, 6, 7
6	5
7	1, 4, 5

How many different storage facilities are necessary in order to keep all seven chemicals?

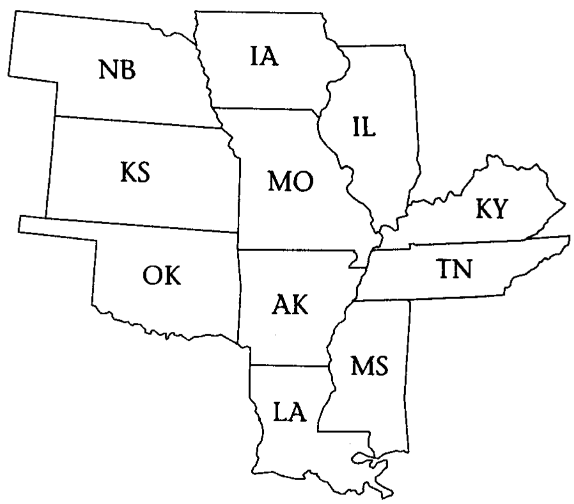
11. Color the following map using only three colors.



12. Draw graphs to represent the following maps. Color the graphs. What is the minimum number of colors needed to color each map?

a.

b.



Projects

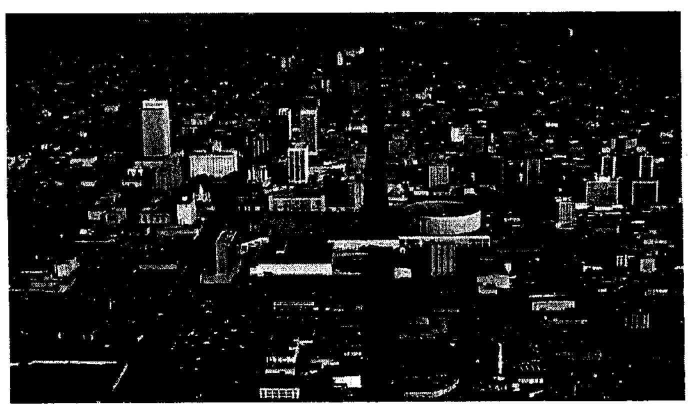
13. Research and report on the mapping industry. Explore questions such as: On the average, how often do maps change? Who are the people/agencies that create maps? How are maps designed and produced? What kind of educational background do cartographers need?

Eulerizing Graphs

Now that you've completed Chapter 4, you have the ability to examine a graph, to determine whether it has an Euler circuit, and to find a circuit if one exists. But, unfortunately, most graphs that represent real-world information do not have Euler circuits. So what does the city street inspector, garbage collector, and utility meter reader do when their street-by-street travels must be made in an optimal manner?

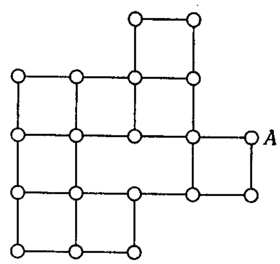
In Lesson 4.4, Exercise 7 (page 180), you solved a problem such as this. The street network graph for the exercise was small, and by careful examination, you were able to find a way to inspect all streets and repeat only a minimal number of them.

This type of problem is often referred to as the Chinese Postman Problem and was first studied by the Chinese mathematician Meigu Guan in 1962. The Chinese Postman Problem differs slightly from the situation that you solved in Lesson 4.4. In the postman problem actual street lengths are examined and attempts are made to minimize the total retraced lengths. In Exercise 7, the assumption was made that all the streets were



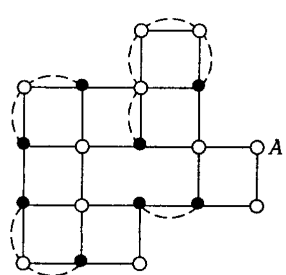
of equal length, so minimizing total length was equivalent to minimizing the number of reused edges.

Consider the following representation of a street network.

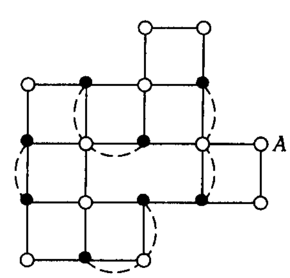


Since this graph has eight vertices of odd degree, there is no way to begin a circuit at *A*, trace each of the edges, and return to *A* without retracing some of the edges. One way to find a circuit that allows for the necessary retracing is to find the degree for each of the vertices and eliminate all vertices of odd degree by connecting them with additional edges. This process is called **eulerizing the graph**. Once the graph is eulerized, an Euler circuit can be found, and the duplicate edges can then be viewed as streets that must be traveled more than once.

What is the "best" way to add the edges in the eulerization process? When adding edges, it is desirable to add ones that duplicate the fewest number of edges in the original graph. If an edge is added that spans more than one existing edge, then possibly a better eulerization can be found. Since only duplicates of edges in the original graph can be added, be careful not to add "new edges." See the following graphs.



One possible eulerization with nine duplicate edges.



A better eulerization with seven duplicate edges.

As you might have guessed, this problem becomes much more difficult as the size of the graph and the number of vertices with odd degree increase. It can be proved that if a graph has n vertices of odd degree, any circuit in the graph that covers every edge at least once must have at least $n/2$ duplications. Be aware that this theorem gives only the lower bound and not the exact number of duplications. For example, in the preceding graph there are eight odd vertices, so at the very least there must be $8/2 = 4$ duplications. In the preceding figure an eulerization was found that has seven retraced edges. The theorem says that it might be possible to do better. Can you find a better eulerization?

If you are interested in knowing more about eulerizing graphs, additional information can be obtained by doing an Internet search or referring to the bibliography at the end of this chapter.

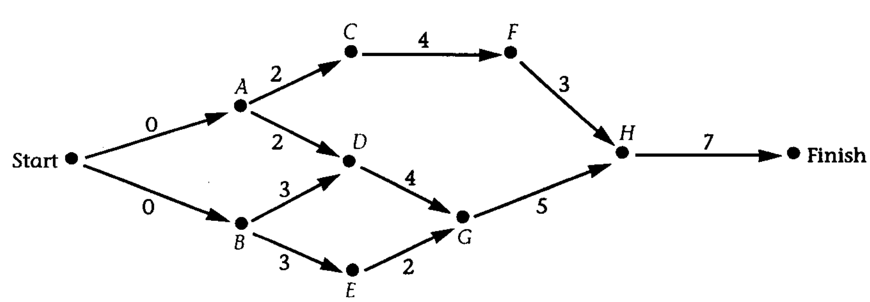
Chapter 4 Review

1. Write a summary of what you think are the important points of this chapter.

2. Draw a graph for the following task table.

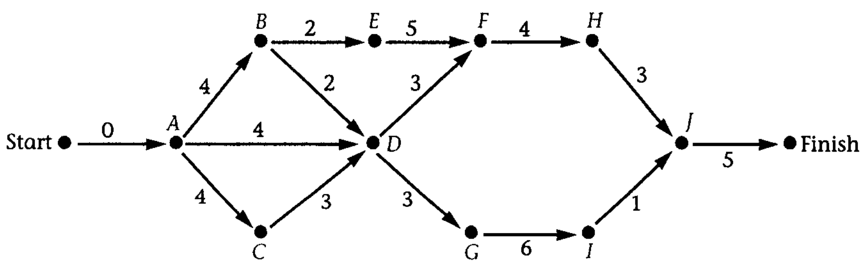
Task	Time	Prerequisites
Start	0	—
A	2	None
B	4	A
C	4	A
D	3	B
E	2	C
F	1	C
G	2	D
H	5	D,E,F
I	3	G,H
Finish		

3. Complete the task table for the following graph.



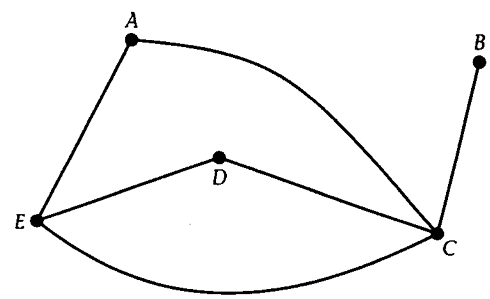
Task	Time	Prerequisites
Start	0	—
A		
B		
C		
D		
E		
F		
G		
H		
Finish		

4. a. List the vertices of the following graph and give their earliest-start time.
b. Determine the minimum project time.



5. Use your graph from Exercise 2.
a. Recopy it and label each vertex with its EST.
b. Determine the critical path and the minimum project time.

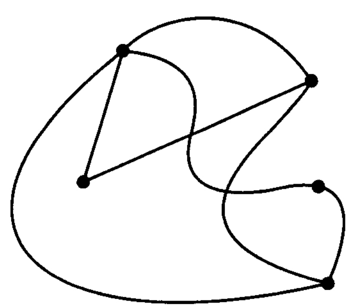
6.



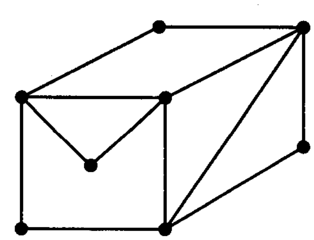
a. Is this graph connected? Explain why or why not.
b. Is this graph complete? Explain why or why not.
c. Name two vertices that are adjacent to vertex E.
d. Name a path from B to E of length 3.
e. What is the degree of vertex C?
f. Determine an adjacency matrix for the graph.

7. Tell whether the following graphs have an Euler circuit, an Euler path, or neither. Explain your answers.

a.



b.



8. Construct a graph for each of the following.

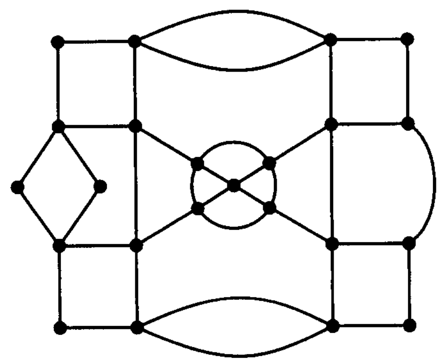
a. $V = \{A, B, C, D, E\}$

$E = \{AE, AB, CD, BC, DE\}$

b.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

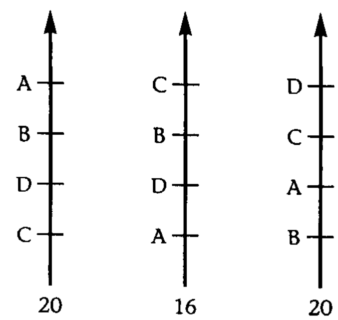
9. Following is a multigraph that represents the downtown area of a small city. The local post office has decided that the mail drop boxes, which are located at the intersection of each street, must be monitored twice daily.



a. Is it possible to find a circuit that begins and ends at the same intersection and visits each drop box exactly once?
b. If not, is there a path that begins at one drop box, visits each drop box exactly once, and ends at a different drop box?
c. If either route exists, copy the figure onto your paper and darken the edges of your proposed route.

10. Use the graph in Exercise 9.
a. Is it possible for the local street inspector to begin at an intersection and inspect each street exactly once?
b. Is it possible for the inspector to finish her route at the same intersection from which she began? Explain why or why not.

11. Consider the following set of preference schedules.



a. Represent this election with a cumulative preference tournament.
b. Is there a Condorcet winner? Explain why or why not.
c. Find several Hamiltonian paths for your graph.
d. Show how to use a Hamiltonian path to construct a pairwise voting scheme (see Exercise 3, Lesson 1.3, page 20) that results in B's winning the election.

12. In scheduling the final exam for summer school at Central High, six different tests must be scheduled. The following table shows the exams that are needed for seven different students.

	Students						
Exam	1	2	3	4	5	6	7
(M) Math	X	—	X	—	X	—	X
(A) Art	—	X	—	X	—	X	—
(S) Science	X	X	—	—	—	—	X
(H) History	—	—	X	—	—	X	—
(F) French	—	—	—	X	X	—	—
(R) Reading	X	X	—	X	X	—	X

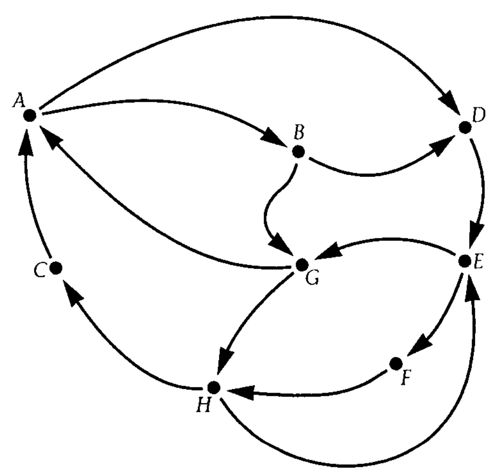
a. Draw a graph that illustrates which exams have students in common with other exams.
b. What is the minimum number of time slots needed to schedule the six exams?

13. The Federal Communications Commission (FCC) is in charge of assigning frequencies to radio stations so that broadcasts from one station do not interfere with broadcasts from other stations. Suppose the FCC needs to assign frequencies to eight stations. The following table shows which stations cannot share frequencies.

Station	Cannot Share with
A	*B, F, H*
B	*A, C, F, H*
C	*B, D, G*
D	*C, E, G*
E	*D*
F	*A, B, H*
G	*C, D*
H	*A, B, F*

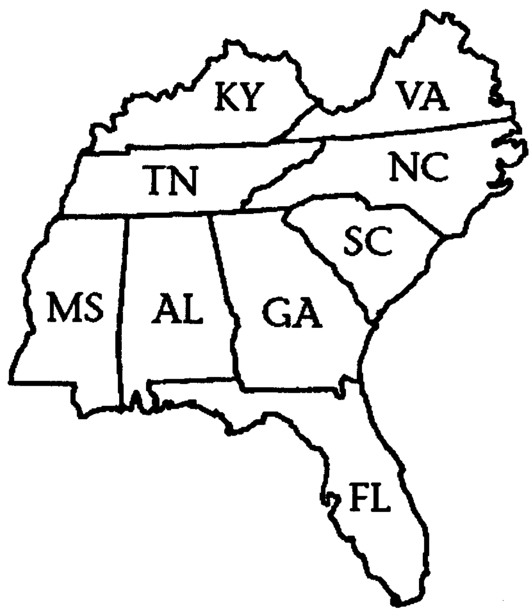
a. Represent this situation with a graph.
b. Find the minimum number of frequencies needed by the FCC.

14. Consider the following digraph.



a. Does it have a directed Euler circuit? Explain why or why not. If it does, list one.
b. Does it have a directed Euler path? Explain why or why not. If it does, list one.

15. a. Represent the following map with a graph.
b. Color your graph.
c. What is the minimum number of colors needed to color the map?



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