CHAPTER

6

Counting and Probability

Lotteries have become quite popular in the United States: consumer spending on lotteries grew at an average annual rate of 11% from 1982 to 1996, reaching a high of about $16 billion in 1996. Each week millions of tickets are sold to people who pick several numbers in hopes that the ones they chose will match those generated by a random process. The lucky few who match several or all of the numbers win anywhere from a few dollars to several million dollars.

In how many ways can a lottery participant choose several numbers from those on a lottery ticket? What is the probability of winning the jackpot in a lottery? How can methods that mathematicians use to determine the probability of winning a gambling game also be used to determine the probability that a medical test's results are correct? How has an understanding of probability improved the reliability of U.S. space shuttle launches?

Lesson 6.1

A Counting Activity

Probability calculations are important in many endeavors. A meteorologist, for example, must calculate the probability of rain, a lottery commission must calculate the probability a player will win, and a medical researcher must calculate the probability that the results of a test are accurate.

Many probability calculations require knowing the number of ways in which an event can happen, such as the number of ways a lottery player can fill out a lottery card. Often the numbers involved are quite large, and careful methods must be used to be sure counting is done properly. However, the best way to begin your work is by considering some situations involving relatively small numbers.

Explore This

The Central High School student council is discussing three fund-raising proposals. Pierre suggests that the council operate a game at the annual

WIZARD OF ID BY BRANT PARKER & JOHNNY HART

school fair. His idea is to write each of the letters of the school's team name, Lions, on a Ping-Pong ball and have participants draw two of the balls from an opaque container. If the letters spell (in the order drawn) a legal word, the participant will win a prize. His proposal has been criticized by some council members who feel it would be too easy to win.

Hilary proposes printing cards with the numbers 1 through 9 displayed in a square matrix and having participants mark two of the numbers (see Figure 6.1). A winning pair would be generated at random, and a prize given to any participant who matches both winning numbers. Her scheme has left council members uncertain about how many winners might be expected in a school of 1,000 students.

1	2	3
4	5	6
7	8	9

Figure 6.1 A card for Hilary's game.

Chuck also wants to operate a game at the school fair. His game would involve a board with the numbers 1 through 6 displayed (see Figure 6.2). A participant would place $1 on any of the numbers, roll two dice, and win a dollar for each time the chosen number appeared. Several council members feel that the organization would lose money on this game.

1	2	3	4	5	6

Figure 6.2 The board for Chuck's game.

Following are three sets of questions related to the games suggested by Pierre, Hilary, and Chuck. As time permits, discuss one or more of the three sets with a few other people.

Here is one way to divide the sets of questions among small groups in your class. At the direction of your instructor, divide your class into groups of three people. Write the numbers 1, 2, or 3 on each of several slips of paper. Have each group draw one of the slips from a bag or box. Each group should consider the set of questions whose number corresponds to the number drawn.

After all groups have finished their discussions, a spokesperson for each group should present the results of the group's discussion to the class. The groups that discussed set 1 should report first, and so forth.

1. Analyze Pierre's proposal. How many different two-letter "words" are there? How many of them are real words? If each of the school's 1,000 students enters exactly once and pays a $1 entry fee, how many winners might there be? How much should each winner receive if the council hopes to raise $500?

> [illegible]
>
> Lottery players in Connecticut ended up breaking the bank over the weekend. On Saturday, thousands of people played the daily lottery and put their money down on number 8-0-0. When that number hit, lottery officials said they had to pay out just over $1 million in winnings, about three times what was wagered.
>
> The state took in $344,277 on the three-digit game and lost $678,170.
>
> The State [illegible] pulls in an average of $340,000 for the daily number and makes a profit of between $170,000 and $270,000.
>
> Of those who bought tickets for Saturday's drawing, 3,026 people bet the "800" number "straight," and another 3,182 chose the number in a three-way box, which means they would win with 800, 080, or 008.
>
> Gamblers who spent 50 cents on a straight bet won $250 while people with three-way bets won $83.50 per bet.

2. Analyze Hilary's proposal. In how many ways could a student fill in the entry form? If each of the school's 1,000 students enters exactly once and pays a $1 entry fee, how many winners might there be? How much should each winner receive if the council hopes to raise $500?

3. Analyze Chuck's proposal. In how many ways could the two dice fall? How often would the council pay the participant $1? $2? How often would the council make $1? Do you think that the council could raise $500 if that is the goal? (If you want to try the game, check with your teacher to see if dice are available in your classroom.)

Exercises

1. One of the goals of this chapter is to develop a few techniques that can be used to determine the number of ways in which an event can happen. The simplest such technique is making a complete list of all possible ways. This is a reasonable method as long as the number of things in the list is not too large. Make a list of all possible "words" that can be made by using two letters of the word *Lions*.

2. Suppose the Ping-Pong balls in Pierre's game are drawn one at a time and the first is kept out of the container while the second is drawn. How many different letters could appear on the first Ping-Pong ball? The second? What is the connection between these numbers and the number of "words" in the list you made in Exercise 1? If the school's team name were Tigers, how many "words" of two letters would there be?

3. Make a list of all possible ways of choosing two numbers from the nine available on one of Hilary's cards. How many are there?

4. If you are filling in one of Hilary's cards, in how many ways could you select your first number? After you've picked your first number, in how many ways could you pick your second number? How are these two numbers related to the number of pairs you listed in Exercise 3?

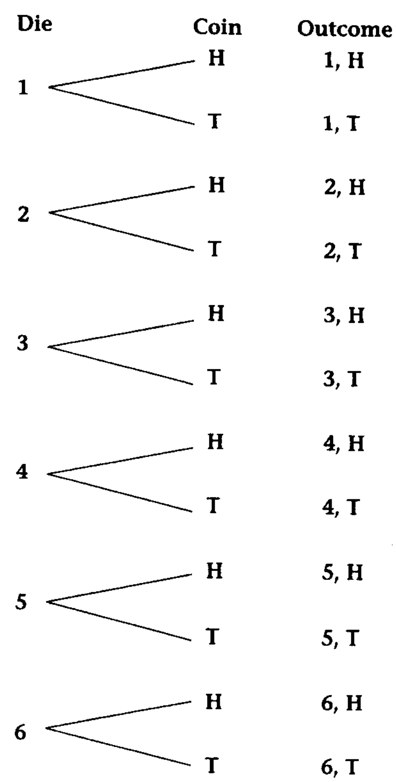
5. Since Chuck's game involves two dice, it is important to be able to distinguish them. Therefore, imagine the dice are two different colors, say red and green. One way the dice could fall is the red die a 3 and the green die a 4. This can be written in shorthand as (3, 4). This outcome is different from the red die a 4 and the green die a 3, which can be written as (4, 3). Make a list of all possible ways the red die and the green die could fall together. How many pairs are in your list?

6. The red die could land in six different ways, as could the green. How are these two sixes related to the number of things in the list you made in Exercise 5?

> **A second counting technique is known in mathematics as the *fundamental multiplication principle*, which says that if events *A* and *B* can occur in *a* and *b* ways separately, then there are $a \times b$ ways that the events can occur together.**

To use the principle, make a blank for each event and write the number of ways each event can occur in a blank. Then multiply these numbers. For example, to determine the number of ways that a die and a coin can fall together, make two blanks: ____ ____, then write the number of possibilities for the die and the coin in the blanks: __6__ __2__, and multiply to get 12.

If a full list of the 12 is needed, a systematic way to ensure that all items are listed is to make a tree diagram like that shown here.



7. Explain how the multiplication principle can be applied to determine the number of different "words" of two letters that can be made from the letters of *Lions*.

8. A utility company in North Dakota once sponsored a contest to promote energy conservation. The contest was to find all legal words that could be made from the letters of *insulate*.
   a. Use the multiplication principle to determine the number of "words" of two letters that are possible.
   b. The multiplication principle can be extended to three or more events. Show how to apply the principle to determine the number of "words" that can be made from three letters of *insulate*.

9. It is possible to modify the multiplication principle to determine the number of ways of selecting two numbers on one of Hilary's cards. Explain how this can be done.

10. Why was it necessary to modify the multiplication principle in Exercise 9?

11. Lotteries often require the participant to select several numbers from a collection of numbers printed on a card. If a state lottery has the numbers 1 through 25 printed in a square matrix, in how many ways can a participant select two of them? Explain.

12. Explain how the multiplication principle can be used to determine the number of ways in which two dice can fall.

13. Make a tree diagram to show all the outcomes when a red die and a green die are tossed together.

14. Make a tree diagram to show all the possibilities when filling out one of Hilary's cards.

15. You are playing Chuck's game and decide to bet on the number 5.
   a. Use the tree diagram you made in Exercise 13 to count the number of ways in which you could win or lose. In how many ways could you win \$1? \$2? In how many ways could you lose \$1?
   b. In the long run, if you played this game many times, do you think you would win or lose money? Explain.

16. In a common carnival dice game, three dice are rolled. Use the multiplication principle to determine the number of ways in which three dice can fall.

17. Counting techniques are useful in many areas other than the analysis of games. An example is genetics. As you may know from your study of biology, a female inherits an X chromosome from her mother and another X chromosome from her father. A male inherits an X chromosome from his mother and a Y chromosome from his father. Use the counting techniques you learned in this lesson to explain the different ways in which chromosomes can be passed from parents to an offspring.

Computer/Calculator Explorations

18. Many calculators have built-in random-number generators that can be modified to simulate random situations. Adapt the random-number generator of a calculator to simulate the games proposed by Pierre, Hilary, and Chuck. Present your work to the members of your class.

Projects

19. Research and report on the impact of lotteries in the United States. What are the benefits and problems associated with lotteries?

Lesson 6.2

Counting Techniques, Part 1

Lesson 6.1 examined several situations in which the answer to the question, In how many ways can this be done? is important. Two important techniques that can be used to answer this question are the multiplication principle and making a complete list. The latter can often be done with the aid of a tree diagram or some other systematic procedure such as an algorithm.

This lesson continues the consideration of counting techniques that began in the last lesson, beginning with the addition principle.

The Addition Principle

Recall that the multiplication principle says that if events A and B can occur in a and b ways, respectively, then events A and B can occur together in $a \times b$ ways. The **addition principle** says that if A and B can occur in a and b ways, respectively, then either event A or event B can occur in $a + b$ ways.

For example, if the student council at Central High consists of 17 members, of which 9 are girls and 8 are boys, and if one girl *and* one boy are to be selected to hold two different offices on the council, then there are $9 \times 8 = 72$ ways of filling the two offices. If a single student, who may be either a boy *or* a girl, is to be selected to hold a single office, then there are $9 + 8 = 17$ ways of making the selection.

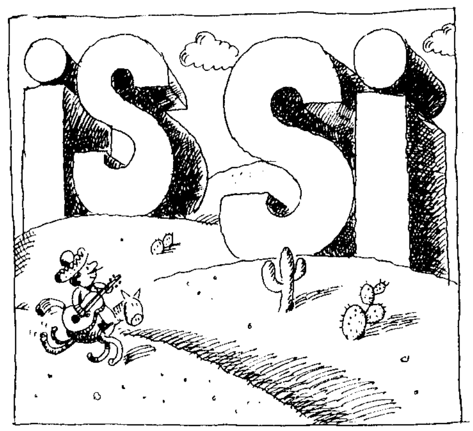
The word *and* in the description of an event often indicates that the multiplication principle should be used, and the word *or* often indicates that the addition principle should be used.

The events "selecting a boy" and "selecting a girl" are called **mutually exclusive** or **disjoint** because a person cannot be both a boy and a girl. On the other hand, events such as "selecting a member of your school's football team" and "selecting a member of your school's basketball team" are not mutually exclusive if there is a person who is a member of both teams. When events are not mutually exclusive, the addition principle requires a modification that you will consider in this lesson's exercises.

Using the Multiplication and Addition Principles Together

The multiplication and addition principles are often used together, as the following example shows. The Central High council members are considering a contest in which words of any length are made from the team name *Lions*.

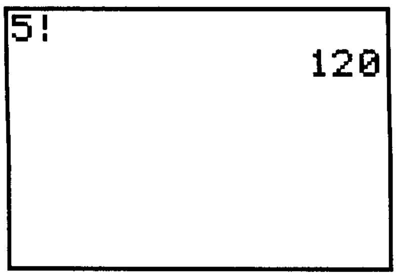
A word of one letter may be composed in only five ways: *l*, *i*, *o*, *n*, and *s*. A word of two letters requires a first letter *and* a second letter, which must be different from the first letter. Thus, there are $5 \times 4 = 20$ ways of composing a word of two letters. A word of three letters requires a first letter and a second letter and a third letter, so there are $5 \times 4 \times 3 = 60$ ways of composing a word of three letters. Similarly, there are $5 \times 4 \times 3 \times 2 = 120$ ways of composing a word of four letters and $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways of composing a word of five letters.



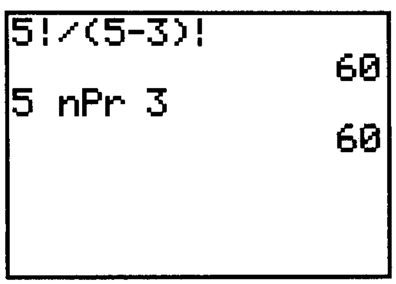
A word may be composed by using one letter *or* by using two letters *or* by using three letters *or* by using four letters *or* by using five letters. Therefore, the total number of words is $5 + 20 + 60 + 120 + 120 = 325$. (Note that the events composing a word by using one letter, composing a word by using two letters, composing a word by using three letters, and composing a word by using four letters are mutually exclusive.)

Factorials, Permutations, and Probability

The calculation of the number of words of five letters than can be made from the letters of *Lions* requires multiplying all integers from 1 through



A factorial calculation on a graphing calculator.



Two ways to calculate a permutation on a graphing calculator.

5. This product is an extension of the multiplication principle and is known as the **factorial** of 5 or, more simply, 5 factorial. A factorial is symbolized by an exclamation mark: 5!. Most calculators have a factorial key or function. If you have never done a factorial on your calculator, try doing so now.

The term **permutation** is often used to describe a counting procedure in which order matters. For example, the game proposed by Pierre is one in which order matters: the words *is* and *si* are not the same. (Situations in which order does not matter are considered in Lesson 6.3.)

Permutations can be computed by using your calculator's factorial key. For example, to find the number of "words" of three letters that can be formed from the letters of *lions*, divide 5! by 2!. Note that this calculation produces the correct result because

$$5 \times 4 \times 3 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}.$$

There are two commonly used symbolic expressions for a permutation of three things selected from a group of five: $P(5, 3)$ or ${}_5P_3$. Either is calculated by evaluating the expression $5!/(5 - 3)!$. Some calculators have a special permutation function.

In general, $P(n, m)$ is calculated by evaluating the expression $\frac{n!}{(n - m)!}$.

Lesson 6.1 began with several questions about the frequency with which certain events could occur. An event's probability is simply the ratio of the number of ways the event can occur to the total number of possibilities in that situation. For example, there are 325 "words" that can be formed from the letters of *Lions*, but only 20 of them have two letters. Thus, the probability of forming a two-letter "word" from the letters of *Lions* is 20/325. Probabilities can be expressed as fractions, decimals, or percentages. As a decimal, the probability of forming a two-letter "word" from *Lions* is .0615, so you would expect a two-letter word about 6 times out of 100.

Because the numerator of a probability is never smaller than 0 and never larger than the denominator, probabilities always range between 0 and 1, inclusive. An event that cannot happen has a probability of 0, and an event that is certain to happen has a probability of 1.

You now have several counting techniques at your disposal:

1. Making a list of all possibilities, for which tree diagrams are often helpful.
2. The multiplication principle and the related factorial and permutation formulas.
3. The addition principle.

Skill at using these techniques develops with practice. The following exercises help develop that skill and also demonstrate some refinements of the three techniques.

Exercises

1. Which is equivalent to $P(10, 4)$, $10!/4!$ or $10!/6!$? Find the value of $P(10, 4)$.

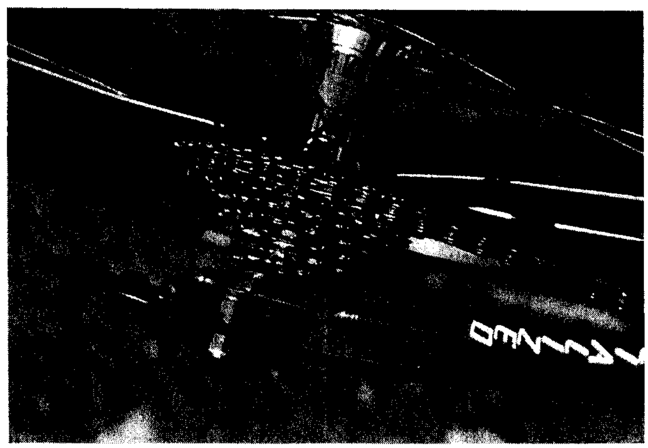
2. At right are the final *USA Today* 1997–1998 season rankings of high school girls soccer teams. If you are a sportswriter voting for the top teams and you can rank only your top 5, in how many ways can you form your ranking from the 25 teams shown?

High School Girls Soccer Rankings

USA TODAY, February 25, 1998

1. Henderson (West Chester, PA) (26-0-0)
2. Winchester (MA) (24-0-0)
3. St. Francis De Sales (Columbus OH) (23-0-0)
4. La Cueva (Albequerque NM) (22-0-0)
5. Massapequa (NY) (18-2-1)
6. Baltimore Catholic (19-1-1)
7. Ramapo (Franklin Lakes NJ) (26-0-0)
8. Jesuit (Portland OR) (20-0-0)
9. Carmel (IN) (23-0-1)
10. Needham (MA) (20-1-3)
11. South Side (Rockville Centre NY) (17-2-1)
12. Governor Thomas Johnson (Frederick MD) (18-1-1)
13. Trumbull (CT) (18-0-1)
14. Santa Rosa (CA) (20-1-0)
15. Mahtomedi (MN) (20-1-2)
16. Roxsbury (Succasunna NJ) (24-1-0)
17. Northport (NY) (16-2-1)
18. Brentwood (TN) (19-1-1)
19. Northfield (VT) (17-0-0)
20. Strath Haven (Wallingford PA) (23-2-1)
21. Brighton (Salt Lake City) (15-1-0)
22. Medina (OH) (19-2-2)
23. Bishop Dennis J. O'Connell (Arlington VA) (18-2-1)
24. Walt Whitman (South Huntington NY) (15-3-1)
25. Simsbury (CT) (16-1-3)

3. A multispeed bicycle has a chain that can be moved to change the bicycle's speed. The rider uses the bicycle's front and rear shift mechanisms to move the chain from one front or rear sprocket to another.



a. If a bicycle has three front sprockets and five rear sprockets, how many speeds does it have?

b. Is it correct to say that a particular speed requires a particular front sprocket *and* a particular rear sprocket, or is it correct to say that a particular speed requires a particular front sprocket *or* a particular rear sprocket?

4. (See Exercise 8 in Lesson 6.1, page 280.)

a. How many different "words" of any length can be made from the letters of *insulate*? (Hint: You can make a word of one letter *or* a word of two letters *or* a word of three letters . . . *or* a word of eight letters.)

b. A group of students is considering entering the contest by programming a computer to print all possible "words" that can be made from the letters of *insulate* and then checking the list against an unabridged dictionary. If the computer prints the words in four columns of 50 words each on a page of paper, how many pages would there be?

5. Some states have vehicle license numbers that consist of three letters followed by three digits. Often the letters *I*, *O*, and *Z* are not used because they can be confused with the numerals 1, 0, and 2, respectively.

a. If these restrictions apply and if characters may be repeated, how many different license plates are possible?

b. What is the probability that a vehicle selected at random will have a license number that begins with *CAT*?

6. a. In how many ways can the coach of a baseball team arrange the batting order of nine starting players?

b. A sportscaster once suggested that a baseball team try every possible batting order for its nine starters in order to determine which one worked best. If a team decides to do so and plays one game each day of the week with a different batting order in each game, how long will it take to complete the experiment?

7. Three math students and three science students are taking final exams. They must be seated at six desks so that no two math students are next to each other and no two science students are next to each other.

a. In how many ways can the students be seated if the desks are in a single row? (Hint: Draw six blanks and use the multiplication principle.)

b. What is the probability that a math student will occupy the first seat in the row?

c. What is probability that math students will occupy the first seat and the last seat?

d. What is the probability that a math student will occupy either the first seat or the last seat?

8. The multiplication principle states that the number of permutations of the letters of the word *math* is 4!. The permutation formula says that $P(4, 4)$ is $\frac{4!}{(4-4)!}$. The denominator of this expression is 0!, which is meaningless. However, in order that 4! and $\frac{4!}{(4-4)!}$ yield the same result, what value must 0! have? Explain.

9. The U. S. Postal Service began using five-digit zip codes in 1963. Every post office was given its own zip code, which ranged from 00601 in Adjuntas, Puerto Rico, to 99950 in Ketchikan, Alaska.

a. If the only five-digit zip code that could not be used was 00000, how many zip codes were possible in 1963?

b. Some five-digit zip codes are prone to errors because they are still legal five-digit zip codes when read upside down. When this happens, a letter goes to the wrong post office and must be returned. How many zip codes are legal when read upside down? (Hint: Draw five blanks, think carefully about which digits could go in each blank, and apply the multiplication principle.)

c. How many of the zip codes you identified in part b are not prone to errors because they read the same when turned upside down?

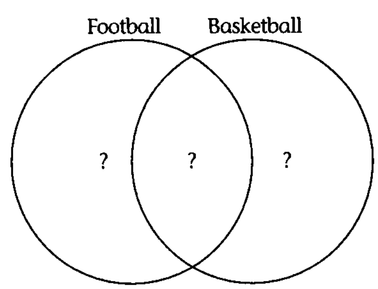
10. a. In how many ways can a person draw 2 cards from a standard 52-card deck if the first card is returned to the deck before the second card is drawn?

b. In how many ways can 2 cards be drawn if the first card is not put back?

11. The addition principle cannot be used as stated in this lesson if the two events are not mutually exclusive. For example, if there are 20 people on your school's basketball team and 45 people on your school's football team, then there are 65 ways of choosing a person from either group only if there are no people on both teams.

a. If there are ten people who play both football and basketball, in how many ways can a person be selected from either team?

b. Write the appropriate number of people in each of the three regions of the following diagram.



c. Describe how the addition principle is applied when two events are mutually exclusive and how it is applied when two events are not mutually exclusive.

d. Event A and event B can occur in a and b ways, respectively, and events A and B have c items in common. Write an algebraic expression for the number of ways in which event A or event B can occur.

e. Central High's soccer team has 38 members, and its basketball team has 13 members. If there are a total of 42 students involved, how many are on both teams. Explain.

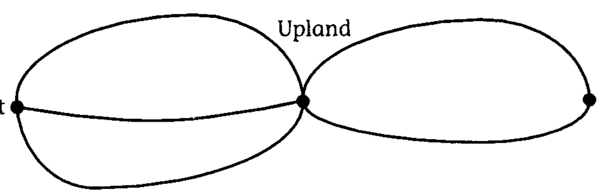
12. Before the 1992 major league baseball season began, Joe Torre, who was then the manager of the St. Louis Cardinals, said he'd picked his starting lineup. He also said he had determined his first three batters but not the order in which they would bat. In how many ways can Joe arrange his batting order if the pitcher must bat last? (Hint: Draw nine blanks and apply the multiplication principle.)

Joe Torre managed the St. Louis Cardinals before becoming manager of the New York Yankees.

13. a. In how many different ways can a teacher arrange 30 students in a classroom with 30 desks?

b. The radius of the earth is approximately 6,370 kilometers, and a standard medical drop is $\frac{1}{10}$ cubic centimeter. Use the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, to find the volume of the earth in drops of water. Compare this with the number of seating arrangements in part a.

14. There are three highways from Claremont to Upland and two highways from Upland to Pasadena.



a. In how many ways can a driver select a route from Claremont to Pasadena?

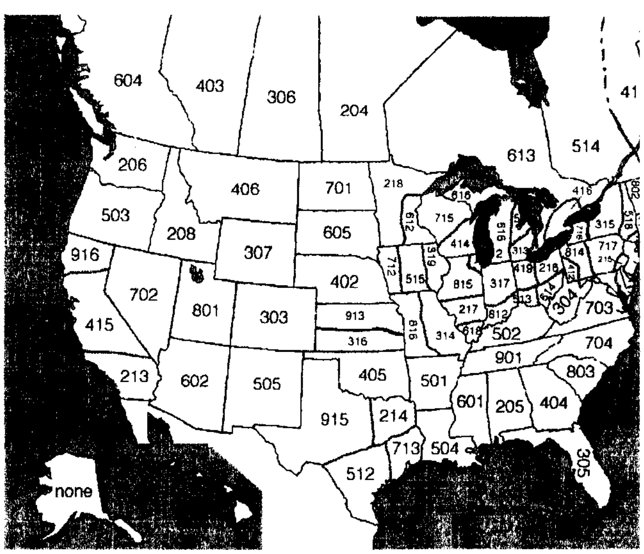
b. Is it correct to say that a trip from Claremont to Pasadena requires a road from Claremont to Upland *and* a road from Upland to Pasadena, or is it correct to say that the trip requires a road from Claremont to Upland *or* a road from Upland to Pasadena?

c. In how many ways can a driver plan a round-trip from Claremont to Upland and back?

The logo for Fordham University radio station WFUV.

15. Radio station call letters in the United States consist of three or four letters, of which the first must be either a *K* or a *W*. Assuming that letters may be repeated, determine the number of radio stations that can be assigned call letters.

16. Six different prizes are given by drawing names from the 72 Central High orchestra members attending the orchestra's annual picnic. In how many ways can the prizes be given if no one can receive more than one prize?



The original area codes.

17. Three-digit telephone area codes were introduced in 1947. At that time, the first digit could not be a 0 or a 1, the second digit could be only a 0 or a 1, and the third could be any digit except 0.

a. How many area codes were possible?

b. Because of shortage of area codes, beginning in 1995 any digit became a legal second digit. How many area codes were possible after 1995?

18. The UPC bar codes consist of two groups of five digits each. One group, as assigned by the Uniform Code Council in Dayton, Ohio, represents the manufacturer, and the other group represents the products of that manufacturer.

0 16000 68790 5

a. How many different manufacturers can be encoded?

b. How many products can each manufacturer encode?

For Many, Area Codes Are Not Created Equal

CHRISTIAN SCIENCE MONITOR, February 6, 1998

Like book ends to the Digital Revolution, residents of New York and San Francisco, and all those in between, will have to get used to different telephone area codes this year.

As of this past weekend, the nice roll-off-your tongue sound of 415 for the San Francisco Bay area has a new companion: 650. And sometime this year, getting ahold of someone in Manhattan won't be as easy as remembering 212.

Driving the need for new area codes is the explosion, particularly over the past two years, in communication needs. Americans bought 15 million pagers in 1996, to accompany the 43 million cell phones already in use. Fax machines have become nearly as common as an office wastebasket. Some 13 percent of the adult population now cruises the Internet. And retail stores started making life easier with machines that required only a swipe of a credit card to pay the bill. What they all have in common is the need for a phone number.

California, for example, had accumulated 13 area codes through its first 50 years of long-distance dialing. During the next two, it will add another 10.

19. Some newspapers publish weekly word puzzles that offer an accumulating cash prize. These puzzles are modified crosswords in which clues are given and usually only two choices are offered.

a. Suppose a prize word puzzle contains 20 questions, each having two possible answers. How many different entries are possible? (Hint: Imagine 20 blanks and use the multiplication principle.)

b. Suppose that you embark on the ambitious project of submitting every possible entry. Further suppose that it takes you 5 minutes to do an entry and that the piece of paper on which you submit an entry is 0.003 inch thick. How long would it take you to prepare the entries, and how thick a stack of paper would you deliver to the newspaper?

20. A carnival game called Chuk-a-Luk is similar to the one proposed by Chuck in the Lesson 6.1, except that three dice are used.
  a. In how many ways can three dice fall? Explain.
  b. Determine the number of ways you could win \$1, win \$2, win \$3, or lose \$1 in the game of Chuk-a-Luk if you win \$1 each time your number shows. (Hint: You win \$2 if the first *and* second dice show your number *and* the third die doesn't, *or* if the first *and* third dice show your number *and* the second die doesn't, *or* if the second *and* third dice show your number *and* the first die doesn't. Draw several sets of three blanks and then use the multiplication and addition principles.)
  c. In the long run, do you think a player would win or lose money in the game of Chuk-a-Luk? Explain.

21. The news article on page 278 describes two types of bets that can be made by selecting three digits from 0 through 9. Discuss the difference in the two types of bets and determine the probability of winning for each type.

22. Factorials can be described recursively. Let $f(n)$ represent $n!$. Write a recurrence relation that expresses the relationship between $f(n)$ and $f(n - 1)$.

Projects

23. Research the history of the study of probability. How did it begin? What roles did Jerome Cardan, Blaise Pascal, and Pierre Fermat play? What problems interested them?

24. Investigate the number of permutations of several objects, of which some look alike. For example, the letters of *math* can be arranged in $4! = 24$ ways; how many different permutations can there be of the letters of *look?* What if there are several sets of identical letters, such as in *Mississippi?* Write a summary that includes a general principle for handling such situations and several examples.

25. Investigate the number of permutations of several objects arranged in circular fashion. For example, Ann, Sean, Juanita, and Herb can be seated along one side of a rectangular table in $4! = 24$ ways. In how many different ways can they be seated around a circular table? Write a summary that includes a general principle for handling such situations and several examples. Explain your interpretation of the meaning of the word *different*.

26. Investigate the use of the addition principle with three events that are not mutually exclusive. Suppose, for example, that the football, basketball, and track teams of Central High have 41, 15, and 34 members, respectively. If 6 people play both football and basketball, 7 are on both the basketball and track teams, 15 are on both the football and track teams, and 4 play all three sports, how many people are involved in one sport or another? Develop a general principle for handling situations of this type and draw a diagram to represent it. Can the principle be extended to four or more events? How?

Lesson 6.3

Counting Techniques, Part 2

The counting techniques developed in Lessons 6.1 and 6.2 are only one method short of forming a fairly complete tool kit for analyzing a variety of probability problems. In this lesson, you consider a technique for counting in situations in which the order of occurrence is unimportant.

Combinations

The game proposed by Hilary in Lesson 6.1 is a simple lottery in which participants must select two numbers from nine printed on a card. The order in which the participant selects the numbers is unimportant. That is, if the winning numbers are 2 and 6, it does not matter whether the owner of a winning ticket selected 2 or 6 first.

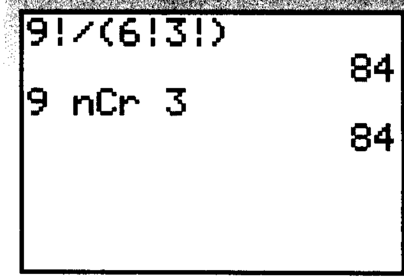
A counting situation in which the order does not matter is called a **combination**. Combinations are counted by modifying the technique used to count permutations. For example, had the order of selection mattered in Hilary's game, then the number of ways of filling out a ticket would be counted as a permutation: $P(9, 2) = \frac{9!}{(9-2)!} = 72$. Because this permutation counts the selection of a pair such as 2 and 6 as different from the pair 6 and 2, every possible pair is counted twice. Thus, the number of combinations of two things selected from a group of 9 is $72/2 = 36$. This combination is expressed symbolically as $C(9, 2)$ or ${}_9C_2$.

If Hilary's game required the selection of three numbers and the order mattered, the number of ways of filling out a card would be $P(9, 3) = 504$. If the order does not matter, then 504 is too large. For example, if the winning numbers are 2, 5, and 8, then 504 counts any arrangement of 2, 5, and 8 as different. The number of ways of arranging 2, 5, and 8 is 3 ×

$2 \times 1 = 6$. Therefore, 504 is six times too large, and $C(9, 3) = \frac{P(9, 3)}{3!} = 504/6 = 84$.

In general, $C(n, m)$ is calculated by evaluating the expression $\frac{P(n, m)}{m!}$. But, $P(n, m) = \frac{n!}{(n - m)!}$, so $C(n, m) = \frac{n!}{(n - m)!m!}$.

Point of Interest



Combinations can be calculated by using a calculator's factorial function or, on some calculators, by using a combination function.

Since there are 36 ways of filling in one of Hilary's lottery tickets, the probability that any one ticket will win is 1/36, or about .028. If 1,000 tickets are sold, Hilary can expect about $1{,}000 \times .028 = 28$ winners. If the game requires the selection of three numbers, the probability a single ticket will win is 1/84, or about .012. If 1,000 tickets are sold, about $1{,}000 \times .012 = 12$ winners can be expected.

Using Combinations with Other Counting Techniques

Combinations are often used along with other counting techniques. For example, the 17-member student council at Central High consists of 9 girls and 8 boys. A committee of 4 council members is being selected. If the positions on the committee are not different in any way, then the order of selection is unimportant, and the number of ways the committee can be selected is $C(17, 4) = \frac{17!}{13!4!} = 2{,}380$.

If the committee must have two girls and two boys, there are $C(9, 2) = \frac{9!}{7!2!} = 36$ ways of selecting the 2 girls and $C(8, 2) = \frac{8!}{6!2!} = 28$ ways of selecting the two boys. Because the committee must consist of 2 girls *and* 2 boys, apply the multiplication principle to conclude that there are $36 \times 28 = 1{,}008$ ways of forming the committee. If the 4 committee members are selected at random, the probability that the committee will consist of 2 girls and 2 boys is $\frac{1{,}008}{2{,}380}$, or about .424.

Now suppose that the committee must consist of either all boys or all girls. There are $C(9, 4) = \frac{9!}{5!4!} = 126$ ways of selecting 4 girls and $C(8, 4) = \frac{8!}{4!4!} = 70$ ways of selecting 4 boys. Because the committee must consist of either 4 girls *or* 4 boys and because all-boy and all-girl committees are mutually exclusive, apply the addition principle to conclude that there are $126 + 70 = 196$ ways of forming the committee. Again, if the 4 committee members are selected at random, the probability the committee will consist of either all boys or all girls is $\frac{196}{2{,}380}$, or about .082.

Exercises

1. Which is larger: $C(10, 2)$ or $C(10, 8)$?

2. Find the sum of all possible combinations of four things. That is, find $C(4, 0) + C(4, 1) + C(4, 2) + C(4, 3) + C(4, 4)$. Do the same for all possible combinations of three things and all possible combinations of five things. On the basis of your results, make a guess about the sum of all possible combinations of six things. Describe any pattern you noticed.

3. In this lesson the number of all-boy four-person committees on the Central High student council was calculated as $C(8, 4) = 70$, the number of all-girl four-person committees was calculated as $C(9, 4) = 126$, and the number of four-person committees that are half boys and half girls was calculated as $C(8, 2) \times C(9, 2) = 1{,}008$.

a. How many four-person committees consist of three girls and one boy?

b. How many committees consist of one girl and three boys?

c. Find the sum of the numbers of committees that consist of four boys, no boys, two boys, three boys, and one boy. Compare this sum with the total number of four-person committees calculated by C(17, 4) in the lesson (see page 295).

4. Darrell Dewey has just left his Central High social studies class and bumped into his friend Carla Cheetham. Darrell informs Carla that Ms. Howe is giving a ten-question true/false quiz today. When Carla asks about the quiz, Darrell says he thought that the quiz was easy and four of the answers were false.

a. When Carla takes the quiz, in how many ways can she select four questions to mark false?

b. In how many ways can Carla select six questions to mark true?

c. In how many ways can Carla fill in the quiz if she ignores Darrel's hint?

5. A standard deck of cards contains 13 different cards from each of four suits: spades and clubs, which are black in color, and diamonds and hearts, which are red in color.

a. In how many ways can 2 cards be dealt from a standard 52-card deck?

b. In how many ways can 2 red cards be dealt from a standard 52-card deck?

c. What is the probability that 2 cards dealt from a standard 52-card deck are both red?

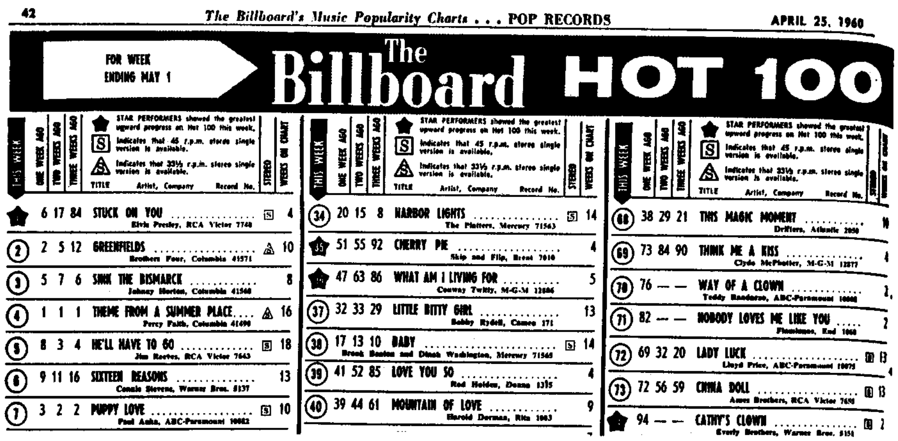
6. Maria has a part-time summer job selling ice cream from a small vehicle she drives through residential areas of her community. She carries six different flavors and sells a two-scoop cone for $1.60.

a. How many two-scoop cones are possible if both scoops are the same flavor?

b. How many two-scoop cones are possible if each scoop is a different flavor?

c. All together, how many two-scoop cones are possible?

7. Hedy Foans, who writes a music column in the Central High Scribbler, has decided to poll students on their favorite songs. She has prepared a list of ten current favorites, from which students will be asked to rank their top three. In how many ways can a student pick a first, second, and third choice from Hedy's ten?



42 *The Billboard's Music Popularity Charts . . .* POP RECORDS APRIL 25, 1960

FOR WEEK ENDING MAY 1 — The Billboard HOT 100

STAR PERFORMERS showed the greatest upward progress on Hot 100 this week.
[S] Indicates that 45 r.p.m. stereo single version is available.
Indicates that 33⅓ r.p.m. stereo single version is available.

THIS WEEK	ONE WEEK AGO	TWO WEEKS AGO	THREE WEEKS AGO	TITLE / Artist, Company, Record No.	STEREO	WEEKS ON CHART
★	6	17	84	STUCK ON YOU — Elvis Presley, RCA Victor 7740	S	4
2	2	5	12	GREENFIELDS — Brothers Four, Columbia 41571	Δ	10
3	5	7	6	SINK THE BISMARCK — Johnny Horton, Columbia 41568		8
4	1	1	1	THEME FROM A SUMMER PLACE — Percy Faith, Columbia 41490	Δ	16
5	8	3	4	HE'LL HAVE TO GO — Jim Reeves, RCA Victor 7643	S	18
6	9	11	16	SIXTEEN REASONS — Connie Stevens, Warner Bros. 5137		13
7	3	2	2	PUPPY LOVE — Paul Anka, ABC-Paramount 10082	S	10
34	20	15	8	HARBOR LIGHTS — The Platters, Mercury 71563	S	14
★	51	55	92	CHERRY PIE — Skip and Flip, Brent 7010		4
★	47	63	86	WHAT AM I LIVING FOR — Conway Twitty, M-G-M 12886		5
37	32	33	29	LITTLE BITTY GIRL — Bobby Rydell, Cameo 171		13
38	17	13	10	BABY — Brook Benton and Dinah Washington, Mercury 71565	S	14
39	41	52	85	LOVE YOU SO — Ron Holden, Donna 1315		4
40	39	44	61	MOUNTAIN OF LOVE — Harold Dorman, Rita 1003		9
68	38	29	21	THIS MAGIC MOMENT — Drifters, Atlantic 2050		[illegible]
69	73	84	90	THINK ME A KISS — Clyde McPhatter, M-G-M 12877		4
70	76	—	—	WAY OF A CLOWN — Teddy Randazzo, ABC-Paramount 10088		[illegible]
71	82	—	—	NOBODY LOVES ME LIKE YOU — Flamingos, End 1068		2
72	69	32	20	LADY LUCK — Lloyd Price, ABC-Paramount 10075	S	13
73	72	56	59	CHINA DOLL — Ames Brothers, RCA Victor 7655	S	13
★	94	—	—	CATHY'S CLOWN — Everly Brothers, Warner Bros. 5151	S	2

8. Ms. Howe has a planter in one of her classroom windows that is divided into five sections. She has purchased two geraniums and three marigolds to plant in the five spaces.

a. In how many ways can Ms. Howe select the two sections in which to plant the geraniums?

b. In how many ways can Ms. Howe select the three sections in which to plant the marigolds?

9. In February 1992, an Australian company sent representatives to Virginia in an attempt to purchase every possible ticket in the state's lottery. The representatives spread their purchases among eight retail chains that had a total of 125 outlets; one representative bought a total of 2.4 million tickets at a single retail chain headquarters. When time ran out, the group had purchased 5 million tickets, or about 70% of all possible combinations.

One of the tickets purchased by the group matched the winning combination of 8, 11, 13, 15, 19, 20. After a controversy over the legality of the purchase, the lottery decided to award the $27 million jackpot to the Australian group, which represented about 2,500 investors who paid an average of $3,000 each. Each investor stood to receive an average of $10,800, at the rate of $540 a year over the 20-year payment period.

a. A Virginia lottery ticket contains the numbers 1 through 44, from which a participant must select six. In how many ways can the selection be made?

b. If it takes 5 seconds to fill out a Virginia lottery ticket, how long would it take one person working 40 hours a week to fill out all possible tickets?

c. If each Virginia lottery form has space for five entries and if each form has a thickness of 0.003 inch, how thick would a stack of forms of all possible entries be?

d. A Florida lottery ticket contains the numbers 1 through 49, from which a participant selects six. In how many ways can this be done?

e. An individual once bought 80,000 tickets in the Florida lottery. What was this person's probability of sharing the $94 million jackpot that had accrued at that time?

f. How does the probability of winning the jackpot in the Virginia lottery compare with the probability of winning the jackpot in the Florida lottery?

[illegible] Foolproof, or [illegible]

[illegible] Albert B. [illegible]

You might think that betting on every possible number combination in a state lottery would guarantee a big payoff.

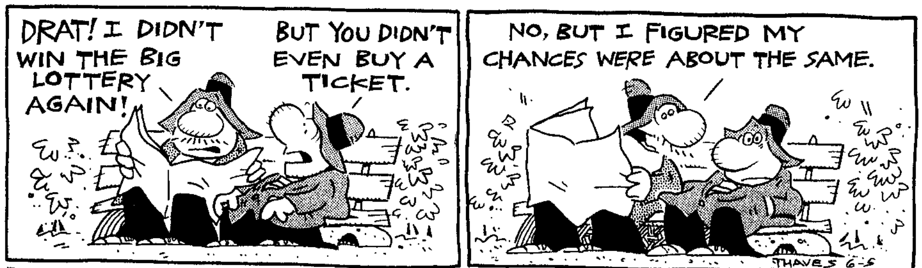
But it doesn't.

A look at what happened in Virginia last week, when an Australian syndicate apparently attempted to corner the lottery, indicates that to be successful, players must not only bet every number, but they must also be lucky enough to be the only winners.

The Australians' system is far from a sure-fire moneymaker, and as soon as more than one [illegible] to be a money-loser. ...

The difference between the size of the investment—around $7 million to cover all the numbers in Virginia—and the $27 million jackpot is not as great as it seems. The grand-prize winnings in most lotteries are paid out over a long time, making them far less valuable than if they were paid all at once.

In fact, after adjusting for inflation and potential investment earnings, $27 million paid out over 20 years is the equivalent of only about $9 million to $12 million if it were paid out today.



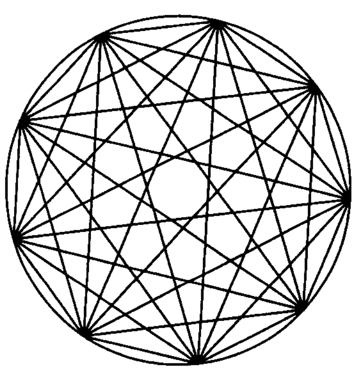
10. Most lotteries include several prizes besides the jackpot. For example, the Virginia lottery gives second prizes to tickets that match 5 of the 6 winning numbers, third prizes to those that match 4 of the 6, and fourth prizes to those that match 3 of the 6.

a. How many different ways are there to receive a second prize? (Hint: The ticket must match 5 of the 6 winning numbers *and* 1 of the 38 nonwinning numbers.)
b. How many different ways are there to receive a third prize?
c. How many different ways are there to receive a fourth prize?

11. Chapter 1 discussed various ways of voting and of determining a winner in elections. Suppose there are seven choices on a ballot.

a. In how many ways can a voter rank the seven choices?
b. Recall that when approval voting is used, the choices are not ranked. In how many ways can you select three choices of which to approve?

12. Dee Noat, the director of Central High's music department, is holding tryouts for the school's jazz band. There are 7 students competing for the three saxophone positions, 8 for the two piano spots, 5 for the two percussion spots, and 12 for the three places as guitarists. In how many ways can Dee select her band?



13. The figure at left was drawn by marking nine equally spaced points on the circumference of a circle and connecting every pair of points.

a. How many chords are there?
b. Number the points from 1 through 9, and explain why drawing the chords is analogous to filling out every possible ticket in Hilary's lottery.

c. Recall that a complete graph is one in which every pair of vertices is connected with an edge. How many edges are there in a complete graph with ten vertices?

14. Emily's Pizza Emporium can prepare a pizza with any one or more of eight available ingredients. In how many different ways can a pizza be ordered at Emily's? (Hint: A pizza can be ordered with one ingredient, or two ingredients, or three ingredients, ...)

15. College Inn Pizza claims that it offers 105 different two-topping pizzas. How many different toppings do you think College Inn Pizza uses? Explain.

16. Carl Burns, coach of the Central High Lions basketball team, has 12 players on his squad. Of these, 3 are centers, 4 are forwards, and 5 are guards.

a. Is it correct to say that a team requires a center *and* 2 forwards *and* 2 guards, or is it correct to say that a team requires a center *or* 2 forwards *or* 2 guards?

b. In how many ways can Coach Burns select his starting team?

17. A telephone exchange consists of all phone numbers with the same three-digit prefix.

a. How many different phone numbers are possible in a given exchange?

b. If a community has 95,000 telephone subscribers, what is the minimum number of exchanges needed?

c. How many phone numbers are possible in a given three-digit area code? (Assume all possible exchanges are permitted.)

18. Allison Gerber, a math teacher at Central High, gives prizes to students in her class who have improved their average grade. At the end of each term she places the names of all qualifying students in a container and draws three.

a. If there are 19 qualifying students and the prizes are three Central High Lions T-shirts, in how many ways can the prizes be awarded?

b. If there are 19 qualifying students and the prizes are a new calculator, a Lions T-shirt, and a book on discrete mathematics, in how many ways can the prizes be awarded?

19. Dominoes come in different-sized sets, of which a double-six set is the most common. In a double-six set, each half of a domino may have any number of spots from 0 through 6. The two halves of a given domino in the set pair a number of spots with itself or with another number of spots.

a. How many dominoes with the same number of spots on each half are there in a double-six set?

b. If every possible pairing is included in the set, how many dominoes with a different number of spots on each half are there in a double-six set?

c. What is the total number of dominoes in a double-six set?

d. Write a description of the way a domino in a double-six set is formed. Explain how the words *and* and *or* in your description reflect the calculations you made to obtain your answer to part c.

e. If you select a domino at random from a double-six set, what is the probability that it will have the same number of spots on each half?

f. How many dominoes are there in a double-twelve set?

20. How many different sums of money can be made from a \$1 bill, a \$5 bill, a \$10 bill, and a \$20 bill? (Hint: You can use one bill at a time or two bills at a time or three bills at a time or four bills at a time.)

21. Many card games involve 5-card hands. (See the description of a standard deck of cards in Exercise 5.)

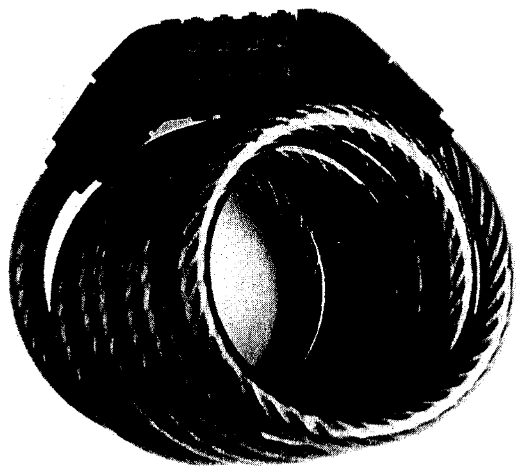
a. How many different 5-card hands can be dealt from a standard 52-card deck?

b. In how many ways can a selection of 3 aces be made from the 4 aces that are found in a standard deck?

c. In how many ways can 3 cards of the same kind (aces, twos, threes, and so forth) be dealt from a standard deck? (Hint: You can deal 2 aces or 3 twos or 3 threes or . . . .)

d. Repeat part c for 2 cards of the same kind.
e. In how many ways can a hand consisting of 3 of one kind and 2 of another (a full house) be dealt from a standard deck?

22. To win the jackpot in the California lottery, a participant must match 6 numbers from 51 that are available. If you buy ten tickets per week in the California lottery, about how often could you expect to win the jackpot? Explain.



23. Some bike locks allow the user to set a four-digit code that opens the lock. These locks are convenient because the user can select a familiar number and is thereby less likely to forget the code.
a. How many different codes are possible?
b. Locks like these are often called combination locks. Do you think this is a good name? Explain.

Projects

24. Research one or more of the lotteries in your part of the country. How large are the jackpots? How many tickets are usually sold? What portion of the proceeds goes to the players? What happens to the rest of the money? Are there any rules to prevent the kind of purchase made by the Australian group in the Virginia lottery? What kinds of strategies are known to be used by players?

25. Investigate probabilities of common card hands. Show how to calculate as many as possible.

Lesson 6.4

Probability, Part 1

The counting techniques discussed in the first three lessons of this chapter can be used to find probabilities of simple events. However, many applications of probability involve compound events that are formed by combining two or more simple events. This lesson is concerned with rules that govern operations on two or more probabilities.

Recall that the probability of an event is the ratio of the number of ways the event can occur to the total number of possibilities. For example, the probability that a die will fall with an even number showing is 3/6 because three of the six possibilities are even. For convenience, the statement "the probability that a die will fall even" is abbreviated *p*(a die will fall even).

The Addition Principle for Probabilities

The addition and multiplication principles you previously used to find the number of ways in which events can occur have counterparts in probability.

> **The addition principle for probabilities states that $p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$, for any two events A and B; $p(A \text{ or } B) = p(A) + p(B)$, for mutually exclusive (disjoint) events A and B.**

The subtraction of $p(A \text{ and } B)$ is unnecessary when events are mutually exclusive because $p(A \text{ and } B) = 0$. When applying the addition principle, do not assume events are mutually exclusive unless you are certain.

Consider Exercise 11 from Lesson 6.2 (pages 288 and 289). In that exercise, 45 people were on a football team, 20 were on the basketball team, and 10 people were on both teams. To determine the total number of people involved, you added 45 and 20, then subtracted 10 to get 55.

If a person is chosen at random from this group, the probability of selecting a football player is $\frac{45}{55}$, the probability of selecting a basketball player is $\frac{20}{55}$; and the probability of selecting someone who is both a football and basketball player is $\frac{10}{55}$. To determine the probability of selecting either a football or basketball player, perform a calculation similar to the one in the previous paragraph: $\frac{40}{55} + \frac{20}{55} - \frac{10}{55} = \frac{55}{55}$, or 1 (see Figure 6.3). Note that without the subtraction of $\frac{10}{55}$, the answer would exceed 1, which is impossible.

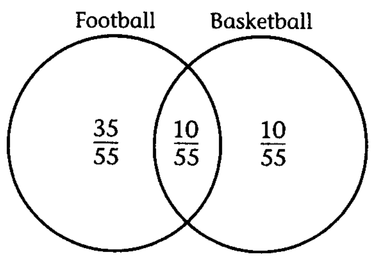


Figure 6.3 The addition principle for probabilities.

The Multiplication Principle and Conditional Probability

The multiplication principle is similar to the addition principle in that it is applied differently depending on the kind of events involved. The addition principle can be shortened if the events are mutually exclusive; the multiplication principle can be shortened if the events are independent.

As an example, consider the following data on the student population at Central High, which has exactly 1,000 students.

	Male	Female	Total
Seniors	156	144	300 (30%)
Juniors	168	172	340 (34%)
Sophomores	196	164	360 (36%)
Total	520 (52%)	480 (48%)	

Many probabilities can be found from these data. For example, the probability of selecting a junior is .34, and the probability of selecting a girl from the junior class is $\frac{172}{340} = .5059$.

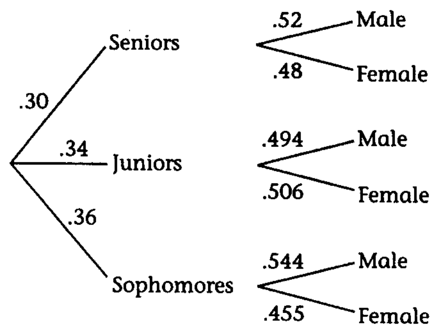


Figure 6.4 Organizing probabilities with a tree diagram.

A tree diagram with appropriate probabilities written along the branches is a convenient way to organize these probabilities (see Figure 6.4).

The probability of selecting a girl from the junior class is called a **conditional probability** because the event describes the selection of a girl under the condition that the person selected be a junior. (Verbal descriptions of conditional events often use the word *from*. Other commonly used words are *if, when,* and *given that.*) The probability of *A* from *B* is sometimes written symbolically as $p(A/B)$.

Note that conditional probabilities change if the order of events is reversed. For example, the probability of selecting a girl from the juniors $\left(\frac{172}{340} = .5059\right)$ is different from the probability of selecting a junior from the girls $\left(\frac{172}{480} = .3583\right)$.

Since 172 of the 1,000 students are girls, the probability of selecting a student who is both a junior and a girl is .172. Refer to the tree diagram in Figure 6.4 and note that the probabilities written along the junior branch and the female branch that follows it are .34 and .506, respectively. The product of .34 and .506 is approximately .172. Thus, the probability of selecting a student who is a junior and a girl equals the product of the probability of selecting a junior and the probability of selecting a girl from the juniors.

The **multiplication principle** for probabilities states that for two events A and B, $p(A \text{ and } B) = p(A) \times p(B \text{ from } A)$. The products that result from applying the multiplication principle can be written to the right of the tree diagram, as shown in Figure 6.5.

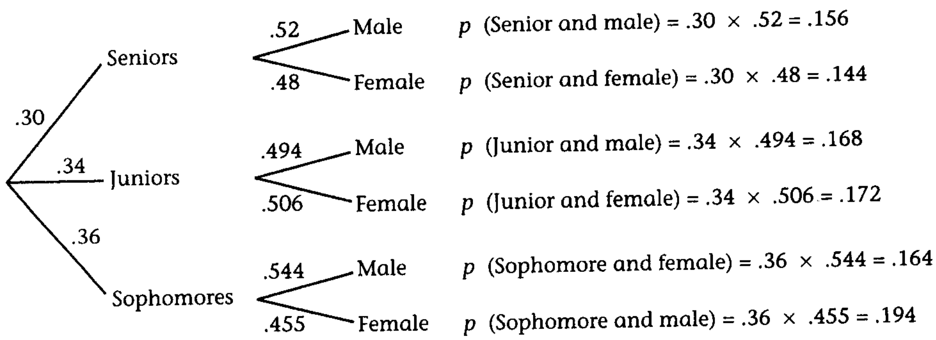


Figure 6.5 The multiplication principle for probabilities.

Refer back to the table and note that 48% of the students are girls. Refer to the tree diagram in Figure 6.4; you'll note something unusual about the senior class: the percentage of seniors who are girls is exactly the same as the percentage of girls in the entire school. For this reason, the events selecting a girl and selecting a senior are called independent.

When events are independent, the probability that one occurs is not affected by the occurrence of the other: If you want to know the probability of selecting a girl, it makes no difference if the selection is made from the entire school or from the senior class. However, the probability of selecting a girl from the entire school (.48) is not the same as the probability of selecting a girl from the junior class $\left(\frac{172}{340} = .5059\right)$. Therefore, the events selecting a girl and selecting a junior are not independent.

> Two events *A* and *B* are **independent** if *p*(*B* from *A*) = *p*(*B*) or if *p*(*A* from *B*) = *p*(*A*). Thus, for independent events, the multiplication principles states that *p*(*A* and *B*) = *p*(*A*) × *p*(*B*).

When applying the multiplication principle, be careful not to assume that events are independent unless you are certain. In some cases, independence is quite obvious. For example, a toss of a coin has nothing to do with the outcomes of previous tosses: the probability that a particular toss is heads is $\frac{1}{2}$ regardless of the previous toss. This independence means that it is correct to calculate the probability of obtaining two heads in a row by multiplying $\frac{1}{2}$ by $\frac{1}{2}$ and obtaining $\frac{1}{4}$.

Just as independence is obvious in some cases, so is the lack thereof. For example, a man who has a beard is more likely to have a mustache than are men in general. Therefore, having a beard and having a mustache are not independent. It would be incorrect to calculate the probability that a man has both a beard and a mustache by multiplying the probability that a man has a beard by the probability that a man has a mustache.

In other cases, it can be difficult to determine whether two events are independent without inspecting data or probabilities. For example, are people who own poodles more or less likely to own pink cars than are people in general? If pink car ownership is either more or less common among poodle owners than among the general public, then the events owning a poodle and owning a pink car are not independent.

There are two ways to determine whether events *A* and *B* are independent:

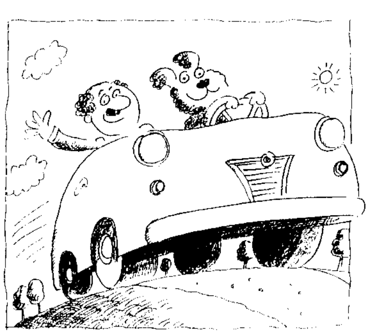
1. Compare *p*(*A*) with *p*(*A* from *B*). If they are the same, *A* and *B* are independent. (You can also compare *p*(*B*) with *p*(*B* from *A*), but it is not necessary to make both comparisons.)

2. Multiply the probability of *A* and the probability of *B*. If the result is the same as the probability of both *A*'s and *B*'s occurring, then *A* and *B* are independent.

Example: Checking for Independence

Consider the following data on ownership of pink cars and poodles in a community. Are owning a pink car and owning a poodle independent in this community?

	Own Poodles	Don't Own Poodles
Own pink cars	250	450
Don't own pink cars	1,250	18,350
Totals	1,500	18,800



To apply the first method of checking for independence, note that the probability of selecting someone who owns a pink car is $\frac{700}{20{,}300} = .0345$, and the probability of selecting someone who owns a pink car from the poodle owners is $\frac{250}{1{,}500} = .167$. Because the two probabilities are not equal, owning a pink car and owning a poodle are not independent.

To apply the second method of checking for independence, note the following probabilities: *p*(owning a poodle) $= \frac{1{,}500}{20{,}300} = .0739$, *p*(owning a pink car) $= \frac{700}{20{,}300} =$

.0345, and p(owning a poodle and owning a pink car) = $\frac{250}{20,300}$ = .0123. Calculate the product of the first two probabilities and compare it with the third: $.0739 \times .0345 = .00255 \neq .0123$.

Attention to detail is important when either the addition or multiplication principles are applied. In this lesson's exercises you use both principles in a variety of settings and consider the consequences of improper application.

Exercises

1. Use the table of Central High student population data on page 305.
   a. What is the probability of selecting a male?
   b. What is the probability of selecting a male from the sophomore class?
   c. Use your answers to parts a and b to determine whether the events selecting a male and selecting a sophomore are independent.
   d. What is the probability of selecting a sophomore?
   e. What is the probability of selecting someone who is both a male and a sophomore? Compare this probability with the product of the probability of selecting a male and the probability of selecting a sophomore.
   f. Multiply the probability of selecting a male from the sophomore class by the probability of selecting a sophomore. Compare this with the probability of selecting someone who is both a male and a sophomore.
   g. Is the probability of selecting a male from the sophomore class the same as the probability of selecting a sophomore from the males?

2. Consider events that describe the performance of your school's football team.
   a. Do you think the events "the team wins" and "the team wins at home" are independent? Explain.
   b. Describe the conditions that would have to exist if the events "the team wins" and "the team wins in bad weather" are independent.

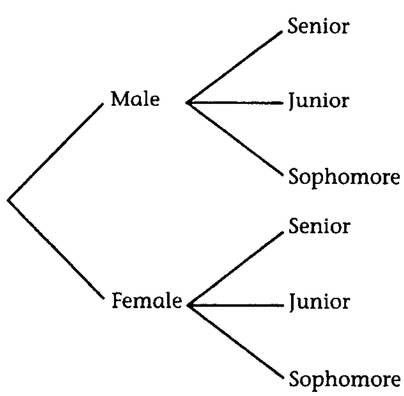
3. Use the table of Central High student population data on page 305.
   a. What is the probability of selecting someone who is either a male or a sophomore?
   b. Add the probability of selecting a sophomore and the probability of selecting a male. Compare this with the probability of selecting someone who is either a sophomore or a male.
   c. Are the events selecting a male and selecting a sophomore mutually exclusive? Explain.

4. Read the *Florida Today* editorial on page 311 about opposition to the Cassini launch.

a. How was the probability $\frac{1}{350}$ obtained?

b. What assumptions were made in the calculation of this probability?

5. The following tree diagram represents the Central High data of this lesson, with the events selecting a male and selecting a female drawn first.



p (Male and Senior) = ___ × ___ = ___

p (Male and Junior) = ___ × ___ = ___

p (Male and Sophomore) = ___ × ___ = ___

p (Female and Senior) = ___ × ___ = ___

p (Female and Junior) = ___ × ___ = ___

p (Female and Sophomore) = ___ × ___ = ___

a. Write the correct probabilities along each branch and complete the calculations at the right.

b. Find the sum of the probabilities you calculated on the right.

6. A card is drawn at random from a standard 52-card deck.

a. What is the probability of drawing an ace?

b. What is the probability of drawing an ace from the diamonds only?

c. Are the events drawing an ace and drawing a diamond independent? Explain.

d. What is the probability of drawing a diamond?

e. What is the probability of drawing a card that is both a diamond and an ace? Compare this probability with the product of the probability of drawing an ace and the probability of drawing a diamond.

f. What is the probability of drawing a card that is either an ace or a diamond?

g. Are the events drawing an ace and drawing a diamond mutually exclusive?

h. Analyze the events drawing a king and drawing a face card (jack, queen, or king). Are they independent? Are they mutually exclusive? Explain your answers.

7. Two cards are drawn separately from a standard deck.

a. What is the probability that the first card is red?

b. What is the probability that the second card is red if the first card is red and is not put back in the deck before the second card is drawn?

c. What is the probability that the first card is red and the second card is red if the first card is not put back in the deck before the second card is drawn?

d. Compare your answer for part c with that for part c of Exercise 5 from Lesson 6.3 (page 297).

e. The following tree diagram represents the colors of cards when two cards are drawn in succession from a standard deck without the first card's being put back. Write the correct probabilities along each branch and complete the calculations at the right.

Cassini Foes Exaggerate Risk of Saturn Mission

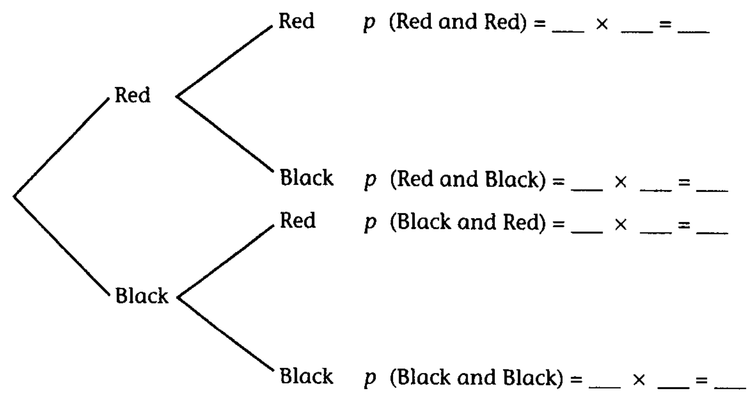
FLORIDA TODAY, October 9, 1997
By Louis Friedman

Washington—A small but vociferous group of antinuclear activists is fighting against Monday's launch of the international Cassini mission, destined for Saturn, because they fear a potential release of plutonium from the on-board power supply. While their concern is understandable, an examination of the issue shows the safety and environmental risks to be very small and the knowledge to be gained very large.

The Cassini spacecraft is designed with a power system that has been employed on 23 planetary missions over the past three decades. It uses plutonium to generate heat, which is converted to electricity to operate the probe. To protect against an accident, the plutonium is encased in special containers that can withstand high impact and temperatures.

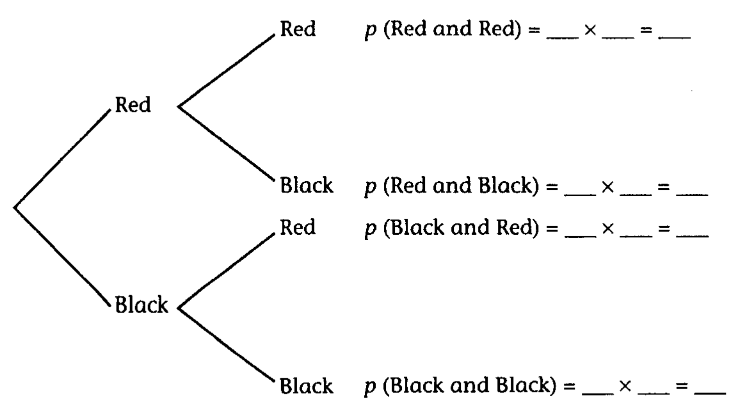
To be sure, plutonium is radioactive and toxic. There is a measurable but small danger from Cassini's plutonium. The probability of an accident during initial launch in which there could be a release of plutonium is 1 in 1,500. The chance that there could be a release in the final launch phase is about 1 in 450. When the probe swings by Earth in a gravity assist two years after launch, the likelihood there could be a release from an accidental re-entry is less than 1 in 1 million.

Put together, the total probability of plutonium release is estimated at about 1 in 350.



f. What is the probability that exactly one of the two cards is red? This occurs if the first card is red and the second card is black or if the first card is black and the second card is red. Find this probability by adding two of the probabilities you calculated and wrote to the right of the tree diagram.

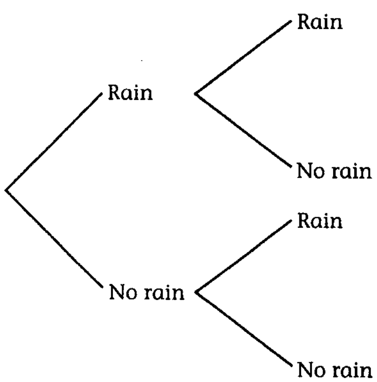
8. Again, two cards are drawn separately from a standard deck.
a. What is the probability that the first card is red?
b. What is the probability that the second card is red if the first card is red and is put back before the second card is drawn?
c. What is the probability that the first card is red and the second card is red if the first card is put back in the deck?
d. The following tree diagram represents the colors of cards when two cards are drawn in succession from a standard deck with the first card replaced before the second is drawn. Write the correct probabilities along each branch and complete the calculations at the right.



e. What is the probability that exactly one of the two cards is red?
f. Is the second draw independent of the first in the situation of Exercise 7 or in the situation of this exercise?

9. The probability that Coach Burns's Central High basketball team will win its first game of the season is .9, and the probability the team will win its second game of the season is .6.
a. What is the probability the team will win both its first and second games?
b. What assumption did you make about the outcomes of the first and second games in answering the previous question? Do you think that this is a safe assumption? Explain.

10. The probability of rain today is .3. Also, 40% of all rainy days are followed by rainy days and 20% of all days without rain are followed by rainy days. The following tree diagram represents the weather for today and tomorrow.



p (Rain and Rain) = ___ × ___ = ___

p (Rain and No rain) = ___ × ___ = ___

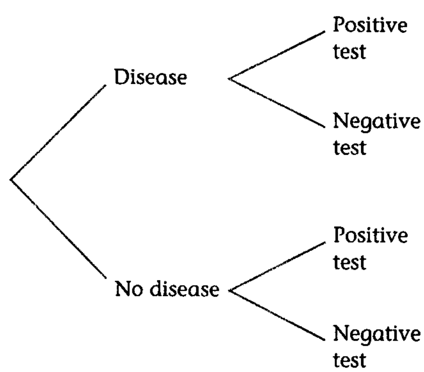
p (No rain and Rain) = ___ × ___ = ___

p (No rain and No rain) = ___ × ___ = ___

a. Write the correct probabilities along each branch and complete the calculations at the right.
b. What is the probability that it will rain on both days?
c. What is the probability that it will rain on one of the two days?
d. What is the probability that it will not rain on either day?
e. Is tomorrow's weather independent of today's? Explain.

11. Some Americans favor mandatory HIV screening for workers in certain professions such as health care. However, medical tests are seldom perfect. When medical tests that are not perfect are used on people who lack symptoms, the results must be carefully interpreted because false positives are common. (A false positive is a positive test result for a person who does not have the disease being tested for.) For example, consider a test that is 98% accurate. That is, it fails to report the existence of the disease in only 2% of those who have it, and it incorrectly reports the existence of the disease in 2% of those who do not have it. About 2 people out of 1,000 have the disease.

a. Write the appropriate probabilities along each branch of the tree diagram and complete the calculations at the right.



p (Disease and Positive test) = ___ × ___ = ___

p (Disease and Negative test) = ___ × ___ = ___

p (No disease and Positive test) = ___ × ___ = ___

p (No disease and Negative test) = ___ × ___ = ___

b. If 100,000 people are screened for the disease, about how many can be expected to test positive for the disease?

c. Of the 100,000 people, how many people who test positive actually have the disease?

d. What is the probability that a person who tests positive for the disease actually has it?

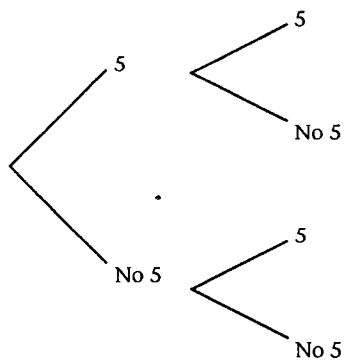
e. Why is the tree diagram in part a ordered the way it is? That is, why would it not make sense to show test results along the first branch and the presence of disease or lack thereof second?

12. Consider the dice game proposed by Chuck in Lesson 6.1 and suppose that you have bet on the number 5.

a. Are the outcomes of the two dice independent of each other? Explain.

b. What is the probability of a 5's appearing on the first die and on the second?

c. The following tree diagram represents the outcomes of the two dice. Write the correct probabilities along each branch and complete the calculations at the right.



p (5 and 5) = ___ × ___ = ___

p (5 and No 5) = ___ × ___ = ___

p (No 5 and 5) = ___ × ___ = ___

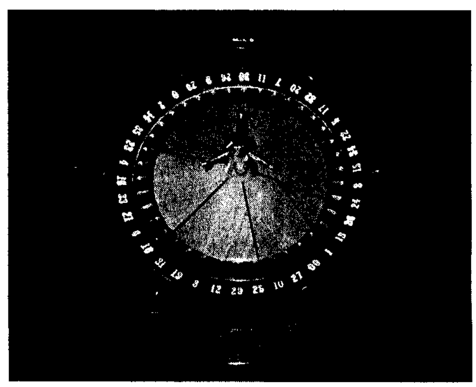
p (No 5 and No 5) = ___ × ___ = ___

d. A single 5 can appear if either the first die shows a 5 or the second die shows a 5. What is the probability that exactly one 5 will appear?
e. What is the probability that no 5s will appear? How is this probability related to the probabilities you found in parts b and d?
f. Do you think you would win or lose money in the long run if you played this game? Explain.

13. The multiplication principle for independent events can be used for more than two such events. The carnival game Chuk-a-Luk, for example, is similar to the game proposed by Chuck in Lesson 6.1, except that three dice are rolled. If you bet on 5, you can calculate the probability that all three dice will show a 5 by cubing the probability that a single die will show a 5. What is the probability that all three dice will show a 5?

14. The probability that one of Ms. Howe's plants will bloom is .9. What is the probability that all five will bloom?

15. European roulette wheels contain the numbers 0 through 36. The ball spun around the perimeter of the wheel has an equal chance of landing on any of these numbers. The British actor Sean Connery once bet on the number 17 three times in a row, and the ball landed on 17 all three times. What is the probability that the ball will land on the same number three times in a row?



Unlike the European roulette wheel, an American wheel has a 00.

Two Columbia O-ring Seals Found Singed

Cape Canaveral, March 5, 1996 (Associated Press)

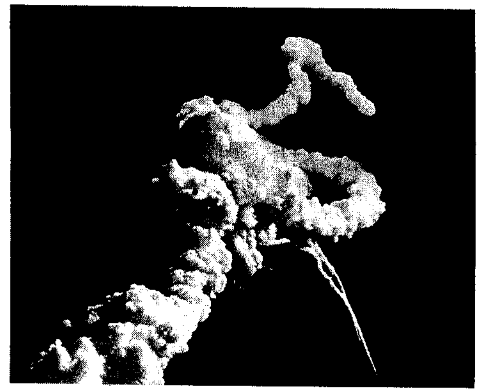
Seals in Columbia's booster rockets were singed during liftoff nearly two weeks ago, the latest in a series of problems with space shuttle O-rings.

The damage, caused by hot rocket gas, posed no threat to Columbia or its seven astronauts, NASA said Monday. Nonetheless, the space agency was treating the problem with ultracaution because of the tragic results of O-ring failure on Challenger 10 years ago.

The problem has been seen [illegible] before, most recently last fall. But it's the first time that gas snaked through the adhesive in more than one place in the so-called case-to-nozzle joint.

Last summer, O-ring seals in different nozzle joints were singed by rocket gas during back-to-back launches. NASA performed extensive repairs before allowing the shuttle to fly again. That problem has not resurfaced.

A leak of hot gas through O-rings in yet another booster joint caused the Challenger to explode in 1986, killing all seven crew members.



This photo was taken just seconds after the space shuttle *Challenger* exploded.

16. On January 28, 1986, the space shuttle *Challenger* exploded over Florida, killing astronauts Greg Jarvis, Christa McAuliffe, Ron McNair, Ellison Onizuka, Judy Resnik, Dick Scobee, and Mike Smith. Many authorities feel this accident could have been prevented if closer attention had been paid to the laws of probability. The rocket that carried the shuttle aloft was separated into sections that were sealed by large rubber O-rings. A presidential commission found that the accident was caused by the leakage of burning gases from the O-rings. Studies found that the probability of a single O-ring's working properly was about .977.

a. If the *Challenger's* six O-rings were truly independent of one another, what is the probability that all six would function properly?

b. After the commission issued its report, some people likened the probabilities in the *Challenger* accident to those in the game of Russian roulette. Compare the probability that all six O-rings will function properly with the probability that a six-chamber revolver will fire if only one chamber contains a bullet and a chamber is selected at random.

c. Each joint was sealed by a system of two O-rings that were supposed to be independent of each other but were not. If the probability that one O-ring fails is $1 - .977 = .023$, what is the probability that both O-rings in a system of two will fail if the two are independent of each other?
d. Subtract your previous answer from 1 to determine the probability that a single joint's seal will work and find the probability that all six will function properly. (Keep in mind that your answer is based on a faulty assumption of independence of the pair of O-rings in a single joint.)

17. It has been estimated that about 1 automobile trip in 100,000 ends in an injury accident.

a. What is the probability that a given automobile trip will not end in an injury accident?
b. If you make approximately 3 automobile trips a day, about how many would you make over a 30-year period?
c. If the outcome of a particular automobile trip is independent of the previous one, what is the probability none of the trips you make over a 30-year period will end in an injury accident?

18. Of the inhabitants of Wilderland, 40% are Hobbits and 60% are humans. Furthermore, 20% of all Hobbits wear shoes and 90% of all humans wear shoes.

a. Make a tree diagram to show the breakdown of residents into Hobbits who either do or do not wear shoes and humans who either do or do not wear shoes. Write the appropriate probabilities along the branches and write the appropriate events and their calculated probabilities to the right of the diagram.
b. Suppose 10,000 residents are selected at random. About how many would you expect to be Hobbits who wear shoes?
c. About how many of the 10,000 would you expect to be shoeless Hobbits?
d. About how many of the 10,000 would you expect to be humans who wear shoes?
e. About how many of the 10,000 would you expect to be shoeless humans?
f. What percentage of the inhabitants who wear shoes are Hobbits?

g. If an inhabitant is selected at random, are any of the events selecting a Hobbit, selecting a human, selecting someone who wears shoes, or selecting someone who doesn't wear shoes independent? Are any of them mutually exclusive? Explain.

19. A witness to a crime sees a man with red hair fleeing the scene and escaping in a blue car. Suppose that one man in ten has red hair and that one man in eight owns a blue car.

a. What is the probability that a man selected at random has red hair and owns a blue car?

b. What assumption did you make when you answered part a?

c. The following table represents the men in the community in question. Inspect the data and present an argument either in favor of or against the assumption.

	Blue Car	No Blue Car
Red hair	1,990	14,010
Nonred hair	18,015	125,985

20. Many games involve rolling a pair of dice.

a. In how many ways can a pair of dice fall?

b. What is the probability of rolling a pair of 6s?

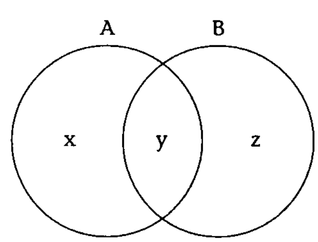
c. What is the probability of rolling a pair of 6s twice in succession?

d. What is the probability of not rolling a pair of 6s?

e. What is the probability of not rolling a pair of 6s twice in succession?

f. A New York gambler nicknamed "Fat the Butch" once bet that he could roll at least one pair of 6s in 21 chances. Find the probability of not rolling a pair of 6s in 21 successive rolls of a pair of dice. (By the way, he lost about $50,000 in the course of several bets.)

21. The circles in the following diagram represent events A and B; the variables x, y, and z represent the number of things in each region. Use



the diagram to explain why the multiplication principle for probabilities: $p(A \text{ and } B) = p(A) \times p(B \text{ from } A)$, is true.

Projects

22. Gather data on your school's student population broken down into the categories of the Central High population in this lesson. Determine whether the data exhibit any events that approximate independence.

23. Research the probabilities of about a dozen real-world events, such as being killed in an automobile accident, winning a lottery in your area, and being killed in a plane accident. Rank them according to their probability of occurring. Do you think the attention given to events in our culture is proportionate to their rate of occurrence? Explain.

24. Real-world data can exhibit events that come close to passing a test for independence. How close is close enough? If you have taken or are taking a statistics course, prepare a report to share with other members of your class that discusses how statisticians decide whether a difference is statistically significant.

25. Research and report on misuses of probability in the courtroom. For example, in a 1968 California case, People v. Collins (68 Cal 2d319), probabilities were erroneously multiplied. What precedent was set by this case? What are other probability-related precedents that have been established?

Lesson 6.5

Probability, Part 2

This chapter began with an examination of several games proposed for a school fund-raiser. The counting and probability techniques you learned in the preceding sections are important tools in analyzing these games and many real-world situations that involve probability. However, the questions of whether the school can expect to make money on these games and how much it can expect to make (or lose) have not been answered completely. This lesson considers two important ideas—probability distributions and expected value—that will enable you to complete your analysis of the games proposed by Pierre, Hilary, and Chuck.

Binomial Probability Distributions

As a first example, consider a brief quiz of three questions in which the answers are either true or false. Because there are only two possible outcomes for any one question, the process of answering a single question is called **binomial**. When the process is repeated several times, however, the multiplication principle states that there are more outcomes than the two that are possible in a single trial. For example, with three trials there are $2 \times 2 \times 2 = 8$ outcomes. A tree diagram can be used as an aid to listing all eight outcomes (see Figure 6.6).

Because each of the outcomes is as likely as any other, the probability associated with each of the eight is $\frac{1}{8}$. The probabilities can also be calculated by writing a probability of $\frac{1}{2}$ on each branch of the tree diagram and multiplying (see Figure 6.7).

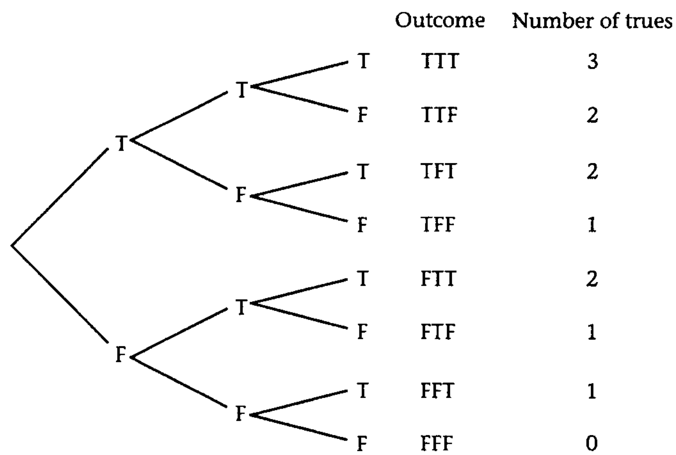


Figure 6.6 A three-question true/false quiz.

The results can be collected into a table called a **probability distribution** table. The way in which the table is constructed depends on whether you are interested in the number of true answers or the number of false answers. If you are interested in the number of true answers, then the table

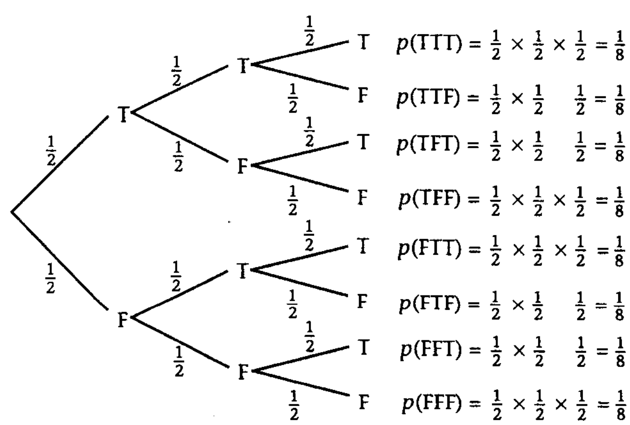


Figure 6.7 Probabilities in a three-question true/false quiz.

looks like this:

Number of True	0	1	2	3
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

A Binomial Probability Shortcut

The tree diagram approach isn't useful if the number of outcomes is large, which is often the case. The counting techniques and the multiplication principle you learned in previous lessons of this chapter can be used to develop an alternative to the tree diagram approach.

For example, consider the probability that exactly two of the three answers are true. The blanks shown below represent the three questions. There are three ways you can select the two to mark true.

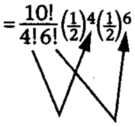
Possibility 1	Possibility 2	Possibility 3
______ ←	______ ←	______
______ ←	______	______ ←
______	______ ←	______ ←

Because the order of selection of the two questions to mark true does not matter, the number of ways of selecting two questions to mark true can be counted as $C(3, 2) = \frac{3!}{2!1!} = 3$. The probability associated with each true answer is $\frac{1}{2}$, and the probability associated with each false answer is also $\frac{1}{2}$, so the probability of two true answers followed by one false answer is $(\frac{1}{2})^2\,(\frac{1}{2})$. This probability is multiplied by 3 because there are three ways two trues can occur.

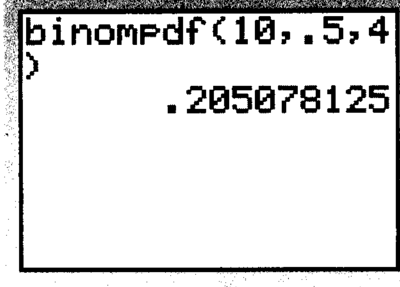
As a second example, consider a quiz of ten questions and the probability that, say, exactly four of them are true. If there are four true, there must be six false, so the probabilities that must be multiplied are four $\frac{1}{2}$s for the true answers and six $\frac{1}{2}$s for the false, which gives $(\frac{1}{2})^4\,(\frac{1}{2})^6$. The number of ways that the four true questions can be selected is $C(10, 4)$. The correct probability, therefore, is $C(10, 4)\,(\frac{1}{2})^4\,(\frac{1}{2})^6 = 210\,(\frac{1}{64})\,(\frac{1}{64})$, or about .205.

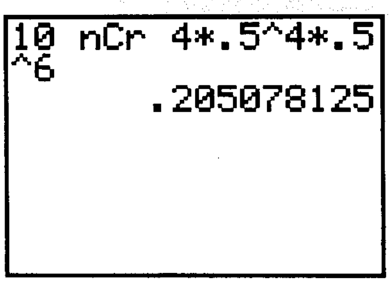
Note that the denominators of the combination formula match the powers of the probabilities.

$$C(10, 4)\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^6 = \frac{10!}{4!6!}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^6$$



Point of Interest





The first screen shows a binomial probability calculation on a calculator with a binominal probability function. The number of trials is the first number, the probability of a single true is the second, and the number of trues is the third. The second screen shows a binomial probability calculation using a calculator's combination function.

When using this method of calculating binomial probabilities, the probability of a single question's being true is multiplied by itself several times. It is essential, therefore, that the individual outcomes be independent of one another. If, for example, the answer to one question depends on an answer to another, then this technique should not be used.

The probability associated with a given outcome is often different from $\frac{1}{2}$. Consider, for example, the dice game proposed by Chuck in Lesson 6.1. Because this game involves only two dice, it is not difficult to analyze it with tree diagrams. However, to serve as an example, the following analysis uses the counting technique just discussed.

If you bet on the number 5 in Chuck's game, then you will win \$2 if two 5s appear, win \$1 if a single 5 appears, or lose \$1 if no 5s appear. Consider the possibility that one 5 will appear. There are C(2, 1) ways to select the die that shows a 5, and the probability of one 5 is $\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$ because one of the dice must be a 5 and the other must be anything but a 5. The correct probability is therefore $C(2, 1)\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^1$, or about .278.

Similarly, the probabilities of no 5s and two 5s can be calculated as $C(2, 0)\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^2$ and $C(2, 2)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^0$, respectively. The probabilities are summarized in the following distribution table.

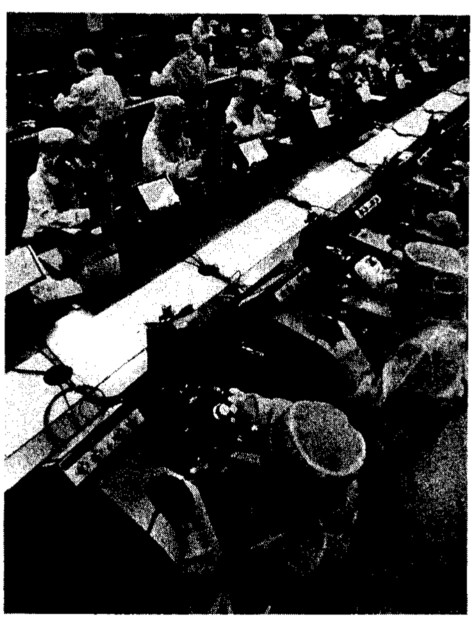
Amount Won	−1	1	2
Probability	.694	.278	.028

In general, if p is the probability associated with a single binomial outcome, the probability of n successes in m attempts is $C(m, n)(p)^n(1 - p)^{m-n}$, provided that individual trials are independent.

Expectation

The question that remains to be answered is how a player could expect to do in Chuck's game in the long run. The player's **expectation** (also known as the expected value of the player's probability distribution) is used to answer this question.

The calculation of expectation for a player in Chuck's game weights each amount that Chuck might win (or lose) according to its probability: $-1(.694) + 1(.278) + 2(.028) = -.36$. The expectation can be interpreted as the average amount the player can expect to win per play of the game. If, for example, the game is played 100 times, the player can expect to be about 100(.36) = \$36 behind.



If the Central High council decides to use this game as a fund-raiser, the council's viewpoint will be the opposite of that of the player: It loses \$2 when two 5s appear, loses \$1 when one 5 appears, and makes \$1 when no 5s appear. Therefore, the council's expectation is +\$.36. The council can expect to make about \$36 for each 100 times the game is played.

A Binomial Probability/Expectation Example

A quality control engineer at the manufacturing plant of an electronics company randomly selects five compact disc players from the assembly line and tests them for defects. If a problem on the assembly line causes the factory to produce 10% defective, what is the probability that the problem will be detected by the engineer's test?

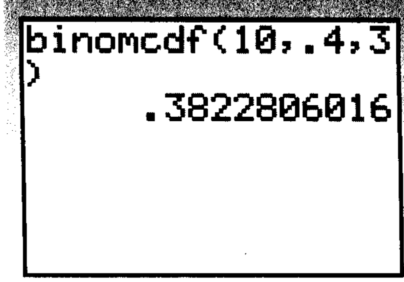
The following table shows the probability distribution for the number of defective players that the engineer found. The probabilities have been rounded to three decimal places.

Number Defective	Probability Calculation	Probability
0	$C(5, 0)(.1)^0(.9)^5$	.590
1	$C(5, 1)(.1)^1(.9)^4$	.328
2	$C(5, 2)(.1)^2(.9)^3$	.073
3	$C(5, 3)(.1)^3(.9)^2$	.008
4	$C(5, 4)(.1)^4(.9)^1$	.000
5	$C(5, 5)(.1)^5(.9)^0$	.000

There is about a 59% chance that the engineer will fail to find a defective player if the defective level is 10%. The probability that the engineer will find at least one defective player is about 41%, which can be found by subtracting 59% from 100% or by adding the last five probabilities in the table.

The engineer's expectation can be calculated as 0(.590) + 1(.328) + 2(.073) + 3(.008) + 4(.000) + 5(.000), or about .498, which can be interpreted as the average number of defective CD players the engineer detects. To put it another way, if the engineer goes through this routine once each day when the defective level is at 10%, the engineer can expect, on average, to detect about one-half a defective CD player a day, or one every two days.

Point of Interest



Some calculators have a function that finds the sum of probabilities in a binomial distribution. This screen shows such a calculation in a situation with a .4 probability of success. The calculated probability is the sum of the probabilities of 0, 1, 2, or 3 successes in 10 trials.

The following exercises treat the use of binomial probability and expectation in a variety of settings.

Exercises

1. Hale Ault, a student at Central High, is known for occasionally neglecting his studies. When he finds a question on an exam that he cannot answer, he uses one of several random processes as an aid. Examine each of Hale's schemes and discuss, first, whether each outcome has the same probability of occurring as does each of the others and, second, whether several successive applications of the scheme are independent of one another.

a. On a true/false question, flip a coin and answer true if the result is heads and false if it is tails.

b. On a three-choice multiple-choice question, flip two coins. Mark the first answer if both coins are heads, the second if both are tails, and the third if the result is one head and one tail.

c. On a four-choice multiple-choice question, associate each of your fingers on one hand with one of the choices, slap your fingers against the desk, and select the one that stings the most.

d. On a four-choice multiple-choice test that allows the use of scientific calculators, use the calculator's random number generator to display a random-number between 0 and 1. Mark the first answer if the number is below 0.25, the second if it is between 0.25 and 0.5, the third if it is between 0.5 and 0.75, and the fourth if it is between 0.75 and 1.

2. Ms. Howe is giving a five-question true/false quiz.

a. In how many ways can a student select three of the questions to mark true?

b. Show how to calculate the probability that exactly three of the questions on the quiz are true.

c. Complete the probability distribution for the number of true answers:

Number of True	0	1	2	3	4	5
Probability						

d. Ms. Howe has a bias toward true answers, and so her questions have true answers about 70% of the time. Recalculate the probability distribution:

Number of True	0	1	2	3	4	5
Probability						

3. Hale Ault is taking a ten-question true/false quiz on which the answers have equal chances of being true or false. Hale is doing the quiz by guessing and needs at least six correct in order to pass.
   a. Find the probability of exactly six correct answers.
   b. Find the probability of exactly seven correct answers.
   c. Find the probability of exactly eight correct answers.
   d. Find the probability of exactly nine correct answers.
   e. Find the probability of exactly ten correct answers.
   f. Hale will pass if he gets six right or if he gets seven right or if he gets eight right or if he gets nine right or if he gets ten right. What is the probability that Hale will pass the quiz?

4. Recall that the game of Chuk-a-Luk is played with three dice and that you win $1 for each time your number shows but lose $1 if your number doesn't show.
   a. Suppose you bet on the number 5. Show how to calculate the probability that 5 shows exactly once.
   b. Complete the probability distribution for the number of times that 5 shows:

Number of 5s	0	1	2	3	4	5
Probability						

   c. Complete the following calculation of your expected winnings: (Note that the distribution of winnings is different from the distribution of the number of 5s because zero 5s results in a loss of $1.)

$$-1(\_\_\_\_\_) + 1(\_\_\_\_\_) + 2(\_\_\_\_\_) + 3(\_\_\_\_\_) = \_\_\_\_\_.$$

   d. If you play the game 100 times, about how much money should you expect to win or lose? Explain.

5. A list of people eligible for jury duty contains about 40% women. A judge is responsible for selecting six jurors from this list.
   a. If the judge's selection is made at random, what is the probability that three of the six jurors will be women?
   b. Prepare a probability distribution table for the number of women among the six jurors.

Convictions in Jay Bias Homicide Reversed

Prosecutor Violated Equal Rights by Blocking Women From Jury

WASHINGTON POST, April 28, 1993, by Jon Jeter

The Maryland Court of Appeals reversed the convictions yesterday of two men in the killing of James S. "Jay' Bias, the brother of the late college basketball star, Len Bias, because the prosecutor blocked more than a dozen women from serving on the jury solely because of their sex.

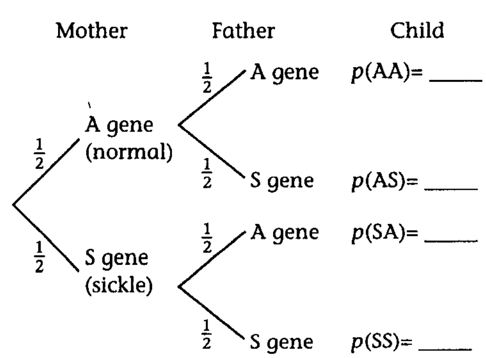
The state's highest court ruled that a Prince George's County prosecutor who tried the murder case against Gerald Eiland and Jerry S. Tyler violated the state's equal rights law by barring the women from serving on the jury. The assistant state's attorney, Mark Foley, used 16 of his 20 peremptory strikes—the legal device by which attorneys from both sides may excuse potential jurors without explanation—to bar women from the jury.

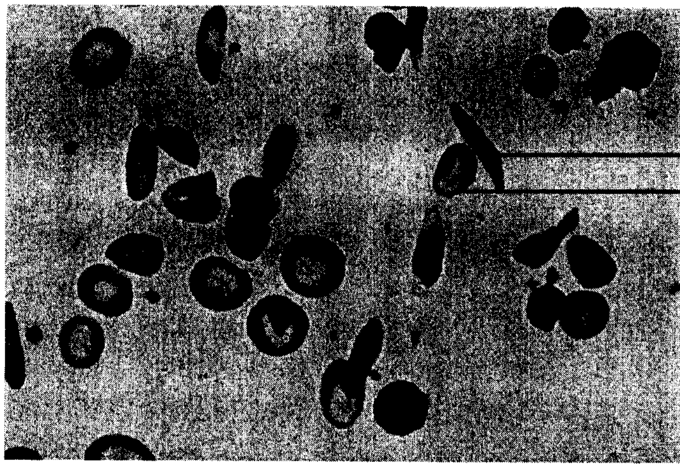
The decision extends to women the protection from discrimination provided by a 1986 Supreme Court decision, which reaffirmed that black people cannot be barred from a jury solely because of their race. And it could end a long-standing practice by some prosecutors who eliminate women from juries based on the belief that they are more compassionate than men and less likely to convict, particularly when the accused is a young man.

c. Suppose that the judge's selection includes only one woman. Do you think this is sufficient reason to suspect the judge of discrimination? Explain.

6. Sickle cell anemia is a genetic disease that strikes an estimated 1 in 400 African-American children in the United States. The disease causes red blood cells to have a crescent shape rather than the normal round shape, which inhibits their ability to carry oxygen. Victims suffer from severe pain and are susceptible to pneumonia and organ failure. Children of parents who are both carriers of the sickle cell gene are frequently stricken.

a. Normal parents have two normal A genes, and carrier parents have one normal A gene and one sickle S gene. A victim of the disease has two S genes. A child inherits one gene independently from each parent. Complete the probability calculations in the tree diagram representing parents who are both carriers.





b. What percentage of children of two carrier parents will have sickle cell anemia? What percentage are carriers? What percentage are normal?

c. A couple who are both carriers have five children. Complete the following probability distribution for the number of children who have the disease.

Number of Children	0	1	2	3	4	5
Probability						

d. Calculate the expectation. Interpret the expectation in this case.

7. A quality control engineer at a widget factory randomly selects three widgets each day for a thorough inspection. Suppose the assembly process begins producing 20% defective.

a. Prepare a probability distribution for the number of defective widgets the engineer will find in the sample of three.

b. Do you think the engineer's quality control scheme is a good one? If not, suggest a way to improve it.

8. A lottery ticket costs $1 and requires you to select 6 numbers from the 44 available.

a. If the $27 million jackpot is the only prize and you do not have to share the jackpot, complete the following probability distribution for your expected winnings (see Exercise 9 in Lesson 6.3, pages 298 and 299).

Amount Won	\$27 million	−\$1
Probability		

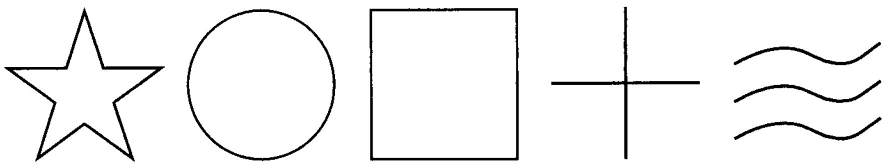
b. Calculate the expected value for this distribution.
c. Recall that an Australian group purchased 5 million tickets in the Virginia lottery. Assume that the jackpot is the only prize, revise the distribution, and recalculate the expectation for the Australian group's winnings.

9. A country has a series of three radar defense systems that are independent of one another. The probability that an enemy plane escapes any one system is .2.
a. Prepare a probability distribution for the number of radar systems that a plane will escape.
b. What is the probability that an enemy plane will penetrate the country's radar defenses?

10. Recall the lottery that Hilary proposed in Lesson 6.1. It required selecting two numbers from the nine available. Hilary has proposed that the price of a ticket be \$1, that the prize for matching both numbers be \$20, and that a prize of \$1 be given to anyone who matches one of the two winning numbers.
a. Prepare a probability distribution for the amount a player can expect to win.
b. Calculate the player's expectation.
c. Would the Central High council make money on this game? Explain. If the council would lose money, suggest a revision of Hilary's plan for awarding prizes so that the council could make money.

11. A fair coin is tossed several times.
a. Find the probability of obtaining exactly 5 heads in 10 tosses. (Do not do the entire probability distribution.)
b. Find the probability of obtaining exactly 10 heads in 20 tosses. Compare this with the previous answer.
c. Prepare a partial distribution of the probabilities of obtaining 4, 5, or 6 heads in 10 tosses.
d. Prepare a partial distribution of the probabilities of obtaining 8, 9, 10, 11, or 12 heads in 20 tosses.
e. Are you more likely to obtain between 40 and 60% heads in 10 tosses or in 20? Explain.

12. Extra Sensory Perception (ESP) is the ability to communicate with another person without speaking. One common test for ESP requires that while one person concentrates on a card selected at random from a special deck, the other person records the image that is received or felt.



a. If the deck consists of five of each of the cards shown here, what is the probability of guessing any one card correctly?
b. Consider a test in which while one person selects a card and concentrates on it, the other person records his or her impression. The card is placed back in the deck, the deck is shuffled, and the experiment is repeated a total of five times. Prepare a probability distribution for the number of cards the receiver can guess correctly.
c. Suppose the receiver gets more than three correct. What is the probability of this happening by chance?

13. Recall the word game proposed by Pierre in Lesson 6.1. Suppose that the only two-letter words made from the letters of *Lions* that are considered legal are *in*, *is*, *on*, *no*, and *so*.
a. What is the probability that a player will draw a legal word from the letters recorded on the Ping-Pong balls?
b. Pierre proposes that the charge for playing the game be $.50 and the prize for selecting a legal word be $1. Prepare a probability distribution for the winnings of someone who plays the game.
c. Calculate the player's expectation.
d. Would the council make money on the game? If your answer is no, suggest a revision of Pierre's plan so that the council could make money.

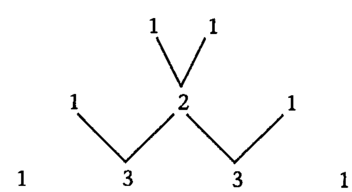
14. Sara Swisher, the Central High Lions' star basketball player, has a field goal percentage of 65%.
a. Sara attempts seven field goals in the first quarter of tonight's game. Prepare a probability distribution for the number of field goals that Sara makes.

b. What assumption have you made? Do you think this is a realistic assumption?
c. Calculate the expectation. What does it mean in this case?

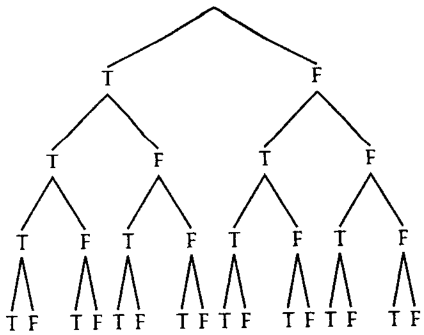
15. Expected value is used to help people make decisions. For example, a person might decide to make an investment if the expectation is positive but not to do so if the expectation is negative.

a. The price of one share of a stock is \$35. You estimate that the probability the price will fall to \$30 is .3, the probability the price will fall to \$25 is .1, the probability the price will increase to \$38 is .4, and the probability the price will increase to \$42 is .2. Should you buy the stock? Explain.
b. A lottery ticket costs one dollar. To win the jackpot, a participant must match 6 numbers from 42 available. Assuming that the participant does not share the jackpot with anyone else and that the jackpot is the only prize, how large must the jackpot be in order for the player's expectation to be positive?
c. Is a positive expected value a good reason to play a lottery? Explain.

16. *Pascal's triangle* is an array of numbers that you may have seen in an algebra class. To construct the triangle, begin with a row of two 1s: 1 1. Each new row starts and ends with a 1, and the other numbers are found by adding the numbers above and on either side of them in this way:



a. Continue the triangle for three additional rows.
b. Calculate all possible combinations of five things: $C(5, 0)$, $C(5, 1)$, $C(5, 2)$, $C(5, 3)$, $C(5, 4)$, and $C(5, 5)$.
c. Where do your answers in part b occur in the triangle?
d. The following tree diagram shows all possible ways of answering true/false quizzes with up to four questions. Fill in the distributions of the number of trues for quizzes of one, two, three, and four questions by tracing the paths of the diagram. Compare these numbers to those in Pascal's triangle.



1 Trial:	Number of true	0	1			
	Number of ways	—	—			
2 Trials:	Number of true	0	1	2		
	Number of ways	—	—	—		
3 Trials:	Number of true	0	1	2	3	
	Number of ways	—	—	—	—	
4 Trials:	Number of true	0	1	2	3	4
	Number of ways	—	—	—	—	—

Computer/Calculator Explorations

17. Write a program for your computer or programmable calculator that does probability distributions of the type discussed in this lesson. The program should accept as input the probability of a single success and the number of trials. It should calculate and display each number of successes and the related probability.

Projects

18. Pascal's triangle contains many patterns other than those in Exercise 16. Investigate and report on some of them.

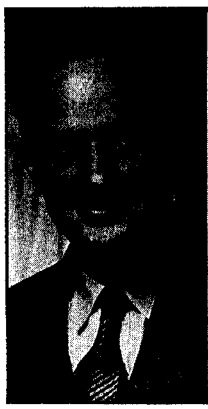
19. Research and report on the use of expected value as a decision-making tool. How, for example, is it used in business?

20. This lesson discusses a combinatorial technique for calculating binomial probabilities. Although useful, this technique has limitations, particularly if the number of trials is very large. Mathematicians sometimes use the Poisson distribution to approximate binomial probabilities. Research and report on the use of the Poisson distribution to approximate binomial probabilities.

21. On September 9, 1990, the Sunday newspaper supplement *Parade* carried a column by Marilyn vos Savant in which a reader posed a problem about the television game show *Let's Make a Deal.* The problem asked whether a contestant should switch doors after the contestant's selection of one of three available doors prompted the show's host to open a door containing a worthless prize. Marilyn's response that the contestant should switch brought a flood of mail, most of which disagreed. Research the controversy and prepare a report on the arguments on both sides. Select the answer with which you agree and defend it.

Monte Carlo Methods

Direct calculation of probabilities of real-world events is sometimes difficult, even for professional mathematicians. Such was the experience of Stanislaw Ulam when he worked in a laboratory in Los Alamos, New Mexico, in the 1940s. Ulam used a computer to simulate the occurrence of events for which he could not calculate probabilities directly. He chose the name "Monte Carlo" to describe his approach.



[illegible]

Stanislaw Ulam came to the United States from Poland in 1935 at the invitation of fellow mathematician John von Neumann. He worked first at the Institute for Advanced Study at Princeton University, then at Los Alamos, where he was instrumental in the development of the hydrogen bomb.

Today's computers are faster and more readily available than those Ulam used at Los Alamos. Even handheld calculators can do many of the tasks only computers could perform a half-century ago.

Consider, for example, the answers to true/false questions. The random selection of a single answer can be simulated by flipping a coin, but the selection can also be simulated by generating a random number. Many calculators have a random-number function that generates random decimals between 0 and 1. The instruction int(2 * rand) doubles this range, then drops the decimal; thus generates random 0s and 1s. (Some calculators have a second random number function that generates random integers

```
0 → T: 0 → F
For (N,1,100)
int(2 * rand) → R
If R = 0
1 + F → F
If R = 1
1 + T → T
End
Disp F,T
```

over a specified range.) A simple program that simulates the random selection of 100 answers to true/false questions is shown on page 335. It runs on most Texas Instruments graphing calculators.

The accuracy of answers obtained with Monte Carlo methods depends on the number of trials that are run: a larger number of trials is more likely to produce an answer close to the correct one than is a smaller number of trials. Thus, you can expect the percentage of trues given by this simulation to be closer to 50 if 1,000 trials are used instead of 100. Computers, it should be noted, can perform a large number of trials much more quickly than calculators.

The following is an example of a difficult problem that can be solved by Monte Carlo methods.

Engineers sometimes need to determine the temperature of a point on a rectangular metal plate from the temperatures at the four edges. Direct calculation requires calculus.

Figure 6.8 represents a 10 centimeter × 6 centimeter plate. Coordinates have been assigned to facilitate the simulation. The point in question is at (6, 4). Edge temperatures are written along the edges.

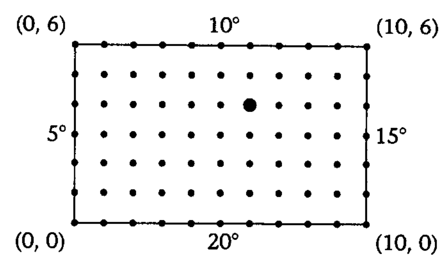
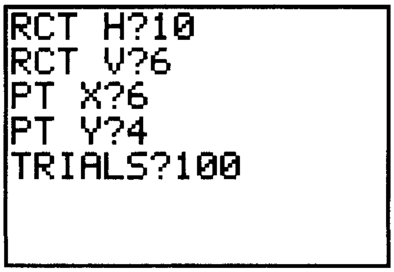
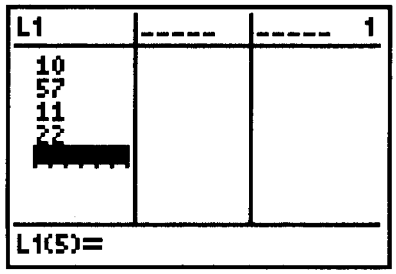


Figure 6.8 A 10 centimeter × 6 centimeter plate.

The Monte Carlo solution to this problem envisions a random walk starting at (6, 4). A random selection determines a movement of either one unit to the right, one unit to the left, one unit upward, or one unit downward. The selection is repeated until the walk terminates at one of the edges. After many trials, the temperature at each edge is weighted according to the portion of trials that terminated at the edge. The weighted average of the four edge temperatures gives an estimate of the temperature at the selected point.





The above screens show the input (left) and results (right) of the TI-83 program RCTRNDWK, which accompanies this book. The user is prompted for the rectangle's horizontal and vertical dimensions, the coordinates of the point, and the number of trials. The numbers of times the walk ended at the left, top, right, and bottom edges are stored, in that order, in the calculator's list L1. Based on these 100 trials, an estimate of the temperature at the point (6, 4) is 5(.10) + 10(.57) + 15(.11) + 20(.22), or about 12°. A more reliable estimate can be made by running 1,000 trials.

Chapter 6 Review

1. Write a summary of what you think are the important points of this chapter.

2. The following table represents ownership of cats among professionals and non-professionals in a community.

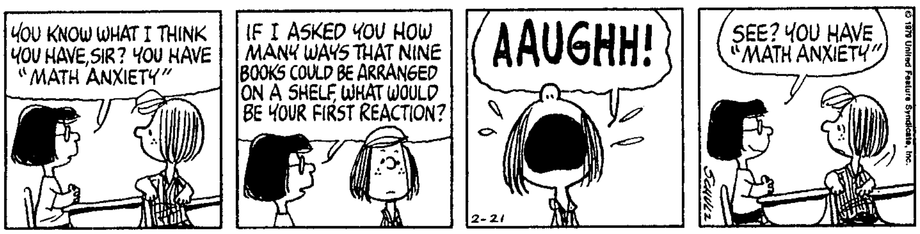
	Own a Cat	Don't Own a Cat
Professionals	2,300	11,400
Nonprofessionals	2,600	27,600
Totals	7,900	39,000

   a. If a person is selected at random from this group, what is the probability of selecting someone who owns a cat?
   b. What is the probability of selecting a cat owner from the professionals?
   c. What is the probability of selecting a professional?
   d. What is the probability of selecting a person who owns a cat and is a professional?
   e. What is the probability of selecting someone who owns a cat or is a professional?
   f. Are the events selecting a cat owner and selecting a professional independent? Explain.
   g. Are the events selecting a cat owner and selecting a professional mutually exclusive? Explain.

3. Lesson 1.5 examined voting situations in which some voters received more votes than others. Recall that a coalition is a collection of voters. In how many ways can you form a coalition of one, two, three, four, or five voters from a group of five voters?

4. Teams A and B are playing in the world series, and each has a 50% chance of winning any game.
   a. What is the probability that team A will win the first four games?
   b. What is the probability that team B will win the first four games?
   c. The series will end in four games if either team A or team B wins the first four games. What is the probability that the series will end in four games?

5. a. In how many ways can six books be arranged on a shelf?
   b. If two of the books are math books, how many arrangements have the math books in the first two positions? (Hint: Draw six blanks and use the multiplication principle.)



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   c. What is the probability that the math books will be in the first two positions?
   d. In how many arrangements are the math books next to each other?
   e. What is the probability that the math books will be next to each other?

6. You are playing a game in which you flip two coins. If both show heads, you win $2; if both show tails, you win $1; but if the coins do not match, you lose $1.
   a. Construct a probability distribution for the amount you could win on a single play of the game.
   b. Calculate your expectation.
   c. If you played the game 100 times, about how much would you expect to win or lose?

7. Being listed first on an election ballot is known to improve a candidate's chances. In order to minimize this effect, ballots could be printed in such a way that each candidate is first on some ballots but not on all. There are three candidates for mayor and five candidates for city council in a local election. How many different orderings are possible if the mayoral candidates must be listed before the council candidates? If 20,000 people vote, about how many would see each ballot?

8. Are you more likely to win the jackpot in a lottery that requires the selection of 5 numbers from 39 or in one that requires the selection of 6 numbers from 36? If 3,000,000 tickets are sold in each lottery, about how many winning tickets would you expect in each?

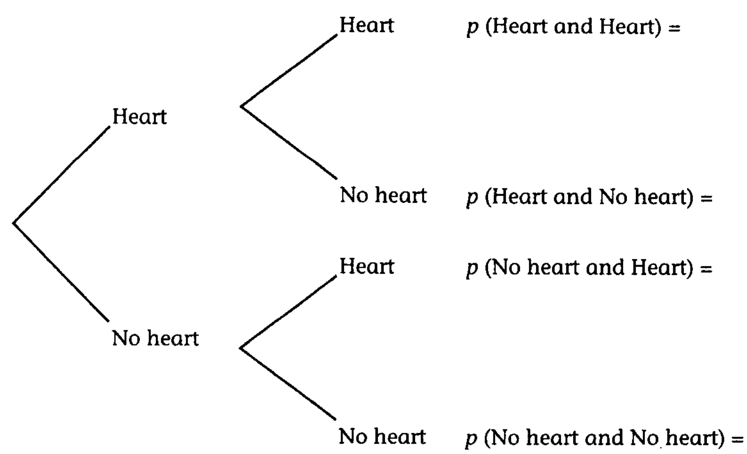
9. Two cards are drawn from a standard deck, and the first card is not put back before the second card is drawn.

a. What is the probability that the first card will be a heart?

b. What is the probability that the second card will be a heart if the first card is a heart?

c. What is the probability the first card will be a heart and the second card will be a heart?

d. The following tree diagram represents the two draws. Write the correct probabilities along the branches and calculate the probabilities at the right.



e. Is the second draw independent of the first? Explain.

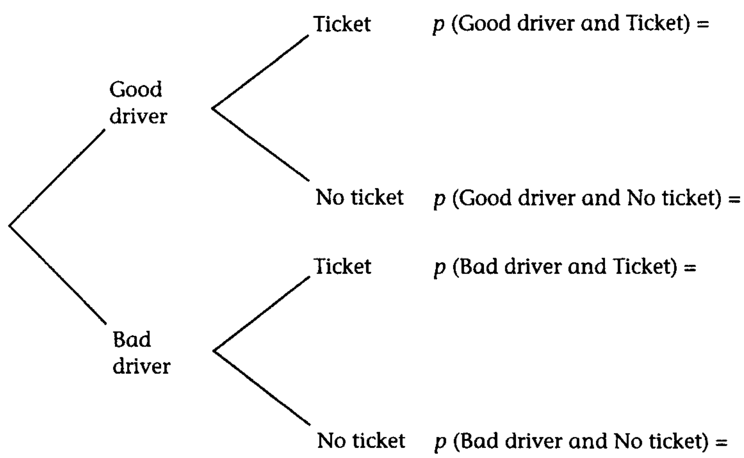
10. There are five boys and six girls in a group, and a committee of three is being selected.

a. In how many ways can the committee be formed?

b. How many committees will consist of one boy and two girls?

c. What is the probability that the committee will have exactly one boy?
d. How many committees will consist of one boy or one girl?
e. What is the probability that the committee will have one boy or one girl?

11. Suppose 90% of all drivers are good, 5% of all good drivers get tickets, and 70% of all bad drivers get tickets.
a. Write the appropriate probabilities along each branch of the following tree diagram and calculate the probabilities shown at the right.



b. If a community has 50,000 drivers, about how many would be expected to get tickets?
c. How many of the people who get tickets are bad drivers?
d. What is the probability that a person who gets a ticket is a bad driver?

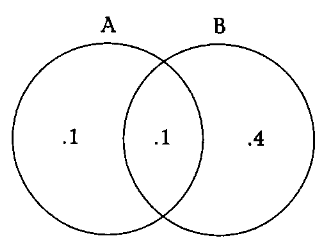
12. Suppose that you and three friends each choose a number between 1 and 10.
a. What is the probability that all three of your friends will pick the same number you pick?
b. What is the probability that all three of your friends will pick a number different from yours?
c. What is the probability that all three of your friends will pick a number that is different from the number picked by any of the others?

13. Two different prizes are being awarded in a group of ten people.
   a. In how many ways can this be done if the same person can win both prizes?
   b. In how many ways can this be done if each person can win no more than one prize?
   c. In how many ways can this be done if each person can win no more than one prize and both prizes are the same?

14. A fair die is rolled five times.
   a. What is the probability that a 6 will appear exactly twice?
   b. What is the probability that a 6 will appear two times or fewer?
   c. What is the probability that a 6 will never appear?

15. Egbert fixes an omelet for breakfast every morning. Depending on what he has in his refrigerator, he adds one or more of the following ingredients to his eggs: mushrooms, green peppers, cheddar cheese, and ham. How many different omelets can Egbert make?

16. The diagram shows the probabilities associated with events *A* and *B*.



   a. What is $p(A)$?
   b. What is $p(A \text{ and } B)$?
   c. What is $p(A \text{ or } B)$?
   d. What is $p(A \text{ from } B)$?
   e. Are *A* and *B* mutually exclusive? Explain.
   f. Are *A* and *B* independent? Explain.

17. The sensitivity (the ability to detect a disease in a person who has it) of a medical test is 90%. The specificity (the ability to detect the absence of a disease in a person who doesn't have it) is 95%. About 400 people in 100,000 have the disease.
   a. If the disease is used to screen a large group of people who lack any signs of the disease, what percentage will test positive?

b. What is the probability that someone who tests positive does not have the disease (a false positive)?
c. In a group of 20 people who are screened, estimate the number who will have false positive tests.

18. A die is rolled. Consider the events the number rolled is divisible by 2, the number rolled is divisible by 3 and the number rolled is divisible by 5. Which pairs of events are mutually exclusive? Which pairs are independent?

Bibliography

Bernstein, Peter L. 1996. *Against the Gods: The Remarkable Story of Risk.* New York: Wiley.

Campbell, Stephen K. 1974. *Flaws and Fallacies in Statistical Thinking.* Englewood Cliffs, NJ: Prentice-Hall.

Casti, John L. 1996. *Would-Be Worlds: How Simulation is Changing the Frontiers of Science.* New York: Wiley.

David, F. N. 1998. *Games, Gods, and Gambling.* Mineola, NY: Dover.

Dewdney, A. K. 1996. *200% of Nothing.* New York: Wiley.

Freedman, David, et. al. 1998. *Statistics.* 2nd ed. New York: Norton.

Green, Thomas A., and Charles L. Hamburg. 1986. *Pascal's Triangle.* Palo Alto, CA: Dale Seymour.

Huff, Darrell. 1959. *How to Take a Chance.* New York: Norton.

Orkin, Mike. 1991. *Can You Win? The Real Odds for Casino Gambling, Sports Betting, and Lotteries.* New York: W. H. Freeman.

Packel, Edward W. 1981. *The Mathematics of Games and Gambling.* Washington, DC: Mathematical Association of America.

Paulos, John Allen. 1988. *Innumeracy: Mathematical Illiteracy and Its Consequences.* New York: Hill and Wang.

Peterson, Ivars. 1997. *The Jungles of Randomness.* New York: Wiley.

Weaver, Warren. 1963. *Lady Luck: The Theory of Probability.* Garden City, NY: Doubleday.

Ulam, Stanislaw M. 1976. *Adventures of a Mathematician* New York: Scribner's.

vos Savant, Marilyn. 1996. *The Power of Logical Thinking.* New York: St. Martin's.