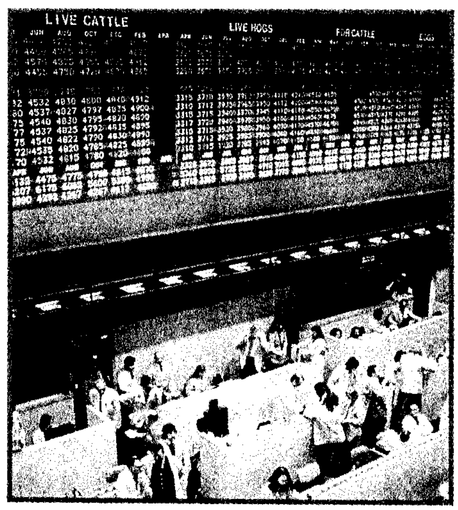
CHAPTER

Matrices Revisited

7

The daily business activity that supplies us with the products and services we need generates large quantities of data that often must be organized into matrices to be understood. Proper organization of these data, as reflected in this board at the Chicago Commodity Exchange, is necessary not only for understanding but also for effective planning.

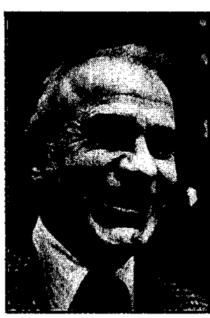
For example, how can a company that provides batteries for another company's compact disc players be sure that it will have enough batteries on hand to fill orders? How does a fast-food chain determine prices that will allow it to do as well as possible against a competitor? How can a meteorologist use data about recent weather activity to predict the weather for tomorrow or a week from now? How can a park service use birth and survival rates of deer in managing herd populations? Matrices demonstrate remarkable versatility in helping to solve these and other real-world problems.



Lesson 7.1

The Leontief Input-Output Model, Part 1

The Leontief input-output model discussed in this lesson is used to analyze the flow of goods among sectors in an economy. This model, developed by Nobel laureate Wassily Leontief of Harvard University in the 1960s, can be applied to extremely complex economies with hundreds of production sectors, such as a country (or even the world), or to a situation as small as a single company that produces only one product.



Mathematician of Note

Wassily Leontief (1906–1999) began his study of input-output analysis in the early 1930s by constructing input-output tables that described the flow of goods and services among various sectors of the economy in the United States. He was awarded the Nobel Prize in economics in 1973 for his work in this area.

You begin your exploration of this model by looking at a simple case. Suppose, for example, that the Best Battery Company of Lincoln, Nebraska, manufactures a particular type of battery that is used to power various kinds of electric motors. Not all the batteries produced by the company, however, are available for sale outside the company. For every 100 batteries produced, 3 (3%) are used by various departments within the company. Thus, if the company produces 500 batteries during a week's time, 15 will be used within the company and 485 will be available for external sales. In general, for a total production of P batteries by this company, $0.03P$ batteries will be used internally and $P - 0.03P$ will be available for external sales to customers.

In other words, the number of batteries available for sale outside the company equals the total production of batteries less 3% of that total production. If D represents the number of batteries available for external sales **(demand)** and P represents the total production of batteries, then the following linear equation can be used to model this situation.

$$P - 0.03P = D. \tag{1}$$

Suppose the company receives an order for 5,000 batteries. What must the total production be to satisfy this external demand for batteries? To find the total production necessary, substitute 5,000 for D in the previous equation and solve for P:

$$P - 0.03P = 5{,}000 \tag{2}$$

$$P(1 - 0.03) = 5{,}000 \tag{3}$$

$$P = 5{,}155 \text{ batteries.} \tag{4}$$

Hence, the company must produce a total of 5,155 batteries to satisfy an external demand for 5,000 batteries.

Notice that the total production of batteries equals the number of batteries used within the company during production plus the number of batteries necessary to fill the external demand.

Now take a look at a simple two-sector economy. Suppose the Best Battery Company buys an electric motor company and begins producing motors as well as batteries. The company's primary reason for this merger is that electric motors are used to produce batteries. Batteries are also used to manufacture motors. The expanded company has two divisions, the battery division and the motor division. It also has changed its name to the Best Battery and Motor Company.

The production requirements of the newly expanded company's two divisions are:

Battery Division

1. For the battery division to produce 100 batteries, it must use 3 (3%) of its own batteries.
2. For every 100 batteries produced, 1 motor (1% of the number of batteries produced) is required from the motor division.

Raytheon Co. to Buy Business-Jet Unit of British Aerospace for $391 Million

Wall Street Journal,
June 2, 1993
by David Stipp and Brian Coleman

Raytheon Co., as expected, agreed to buy British Aerospace PLC's business jet unit for about $391 million.

The planned purchase would help Raytheon move beyond its core defense business and enable British Aerospace to reduce debt and better focus on its principal businesses, including automobiles and defense.

The purchase would nearly double Raytheon's corporate-jet market share and increase its overall aircraft sales to about $1.7 billion from $1.3 billion annually. Raytheon owns Beech Aircraft, a Wichita, Kan., maker of turboprop and jet aircraft, which has about 12% of the market for light to medium jets. The industry leader for business jets, Textron Inc.'s Cessna Aircraft unit, has an estimated 60% market share.

Max Bleck, Raytheon's president, said that although the small-plane market "has been in general malaise," Raytheon is positioning itself for an industry turnaround by buying when acquisition prices are low. Mr. Bleck reiterated Raytheon's goal of boosting to 50% from about 30% its portion of operating profit that is derived from non-defense business. He said Raytheon remains interested in possible acquisitions in the engineering, construction and appliance areas.

Motor Division

1. For the motor division to produce 100 motors, it must use 4 (4%) of its own motors.
2. For every 100 motors produced, 8 batteries are required from the battery division. (Notice that the number of batteries required is 8% of the total number of motors produced.)

The production needs within this two-sector economy can be represented visually by using a weighted digraph as shown in Figure 7.1. The digraph shows that if the company needs to produce a total of b batteries and m motors, then:

1. The battery division will require $0.03b$ batteries from its own division and $0.01b$ motors from the motor division.
2. The motor division will require $0.04m$ motors from its division and $0.08m$ batteries from the battery division.

A third way to present the information regarding the company's production needs is with a matrix. This matrix, which shows the required input from each sector of the economy, is called a **consumption matrix** for the economy.

$$\begin{array}{ccc} & & \text{To} \\ & & \begin{array}{cc}\text{Battery} & \text{Motor}\end{array} \\ \text{From} & \begin{array}{c}\text{Battery} \\ \text{Motor}\end{array} & \begin{bmatrix} 0.03 & 0.08 \\ 0.01 & 0.04 \end{bmatrix} \end{array}$$

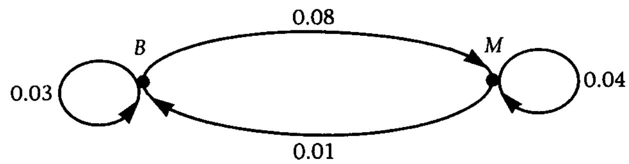


Figure 7.1 Weighted digraph showing the input required by each division.

As an example, to produce 200 batteries, the battery division will need (0.03)(200) = 6 batteries from the battery division and (0.01)(200) = 2 motors from the motor division.

Explore This

At the direction of your teacher, divide your class into groups of three to four people. Each group is to represent a production management team for the newly merged battery and motor company just described.

The company has given the battery division a daily total production quota of 1,000 batteries. The daily quota for the motor division is 200 motors. The production management team's problem is to determine how many batteries and motors each of the company's divisions will need during production and how many batteries and motors will be available for sales.

To solve the problem, the management team must complete the following tasks:

1. Find the number of batteries and motors needed by the battery division to meet its quota. Repeat for the motor division.

2. Find the number of batteries available for external sales if both divisions meet their daily quotas. Repeat for the number of motors available for sales outside the company.

3. Make up a formula for computing the number of batteries that the company will use internally to meet its production quotas. Repeat for the number of motors used internally. These formulas will be used to program a computer to do future calculations to save the management team some work. Let b represent the total number of batteries produced, and m represent the total number of motors produced by the company.

4. Explore this problem from another point of view. Suppose the company needs to fill an order of 400 batteries and 100 motors. Estimate the total production needed from each division to fill these orders. Check your estimate and revise it if necessary. Hint: Recall that the amount of a product available to fill an external order (outside demand) equals the total production minus the amount of that product consumed internally by the company.
5. If time permits, find a system of two equations in two unknowns that could be used to find the total production for each division required in task 4.

After all groups have finished tasks 1 through 4, a spokesperson for each production management team should present the results of the team's discussion to the class.

Exercises

1. A utility company produces electric energy. Suppose that 5% of the total production of electricity is used up within the company to operate equipment needed to produce the electricity. Complete the following production table for this one-sector economy.

Total Production Units	Units Used Internally	Units for External Sales
500	.05(500) = —	500 − .05(500) = —
900	—	—
—	100	—
—	250	—
—	—	2,375
—	—	7,125
P	—	—

2. Suppose that for every dollar's worth of computer chips produced by a high-tech company, 2 cents' worth is used by the company in the manufacturing process.

a. What percentage of the company's total production of computer chips is used up within the company?

b. What would the weighted digraph look like for this situation?

c. Write an equation that represents the dollars' worth of computer chips available for external demands (D) in terms of the total production (P) of chips by the company.
d. What must the total production of computer chips be in order for the company to meet an external demand for \$20,000 worth of computer chips?

3. The high-tech company described in Exercise 2 adds another division that produces computers. Each division within the expanded company uses some of the other division's product:

Computer Chip Division: Every dollar's worth of computer chips produced requires an input of 2 cents' worth of computer chips and 1 cent's worth of computers.

Computer Division: Every dollar's worth of computers produced requires an input of 20 cents' worth of computer chips and 3 cents' worth of computers.

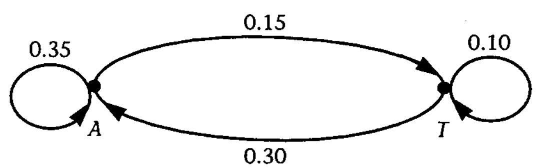
a. Draw a weighted digraph that summarizes the production needs for this two-sector economy.
b. Construct a consumption matrix for this economy.
c. Suppose the computer chip division produces \$1,000 worth of chips. How much input does it need from itself and from the computer division?
d. Suppose the computer division produces \$5,000 worth of computers. How much input does it need from itself and from the computer chip division?
e. What must the total production of computer chips be for the company to meet an external demand for \$25,000 worth of computer chips?
f. What must the total production of computers be for the company to meet an external demand for \$50,000 worth of computers?

4. A company has two departments: service and production. The needs of each department within this company (in cents per dollar's worth of output) are shown in the following consumption matrix.

$$\begin{array}{c} \\ \text{Service} \\ \text{Production} \end{array} \begin{array}{c} \begin{array}{cc} \text{Service} & \text{Production} \end{array} \\ \begin{bmatrix} 0.05 & 0.20 \\ 0.04 & 0.01 \end{bmatrix} \end{array}$$

a. Draw a weighted digraph to show the flow of goods and services within this company.
b. Complete the following: For every dollar's worth of output, the service department requires ______ cents' worth of input from its own department and ______ cents' worth of input from the production department. For every dollar's worth of output, the production department requires ______ cents' worth of input from its own department and ______ cents' worth of input from the service department.
c. Suppose that the total output for the service department is $20 million over a certain period of time. How much of this amount is used within the service department? How much input is required from the production department?
d. The total output from the production department is $40 million over the same time period. How much of this total output is used within the production department? How much input is required from the service department?
e. Combine the information in parts c and d to find how much of the total output from the service and production departments will be available for sales demands outside the company.

5. The weighted digraph below represents the flow of goods and services in a two-sector economy involving transportation and agriculture. The numbers on the edges represent cents' worth of product (or service) used per dollar's worth of output.



a. Construct a consumption matrix for this situation.
b. Complete the following: For every dollar's worth of output the agriculture sector requires ______ cents' worth of input from its own sector and ______ cents' worth of input from the transportation sector. For every dollar's worth of output the transportation sector requires ______ cents' worth of input from its own sector and ______ cents' worth of input from the agriculture sector.
c. If the total output for the agriculture sector is $50 million dollars and the total output for the transportation sector is $100 million dollars over a certain period of time, find:

i. The total amount of agricultural goods used internally by this two-sector economy.
ii. The amount of agricultural goods available for external sales.
iii. The total amount of transportation services used internally by this two-sector economy.
iv. The amount of transportation services available for external sales.

d. Write an equation to represent the internal consumption of agricultural products for this economy. Let P_A represent the total production for agriculture and P_T represent the total output for transportation.
e. Write an equation to represent the internal consumption of transportation services. Use P_A and P_T as in part d.
f. Using the information from parts d and e, write equations representing the amount of agricultural products and transportation available to fill external consumer demands. Let D_A represent the total available for external demand for agriculture and D_T represent the total available for external demand for transportation. Recall: The amount of a product available to fill external demands is equal to the total production less the amount that is used internally.
g. Suppose that this economy has an external demand for \$10 million dollars' worth of agricultural products and \$15 million dollars in transportation services. Write a system of equations that could be solved to find the total production of agriculture products and transportation services necessary to satisfy these demands.

6. Complete task 5 for the production management team of the Best Battery and Motor Company (see page 350).

Project

7. Research and report to the class on the life and work of Wassily Leontief.

Lesson 7.2

The Leontief Input-Output Model, Part 2

As you can see from the exercises in Lesson 7.1, the Leontief model becomes quite complicated very quickly, even when dealing with only one- and two-sector economies. You are probably thinking that there has to be an easier way. You're right. There is an easier way.

Go back and review your Best Battery and Motor Company exploration in Lesson 7.1 (page 349). The production needs of each of the two sectors are summarized in the consumption matrix (C) for this economy.

$$C = \begin{array}{r} \\ \text{Battery} \\ \text{Motor} \end{array} \begin{array}{c} \begin{array}{cc} \text{Battery} & \text{Motor} \end{array} \\ \left[\begin{array}{cc} 0.03 & 0.08 \\ 0.01 & 0.04 \end{array} \right] \end{array}$$

In Exercise 6 (page 353) you were asked to find a system of two equations in two unknowns that could be solved to find the total production of batteries (b) and motors (m) necessary to meet external sales demands of 400 batteries and 100 motors. To find two such equations, you used the fact that the total production for each product equaled the amount of the product used up by the two divisions during production plus the external demand for that product (sales outside the company).

For example, the total number of batteries produced (b) equals the number of batteries used within the battery division ($0.03b$) plus the number of batteries sent to the motor division ($0.08m$) plus the number of batteries required for sales outside the company (400). This gives the equation $b = 0.03b + 0.08m + 400$.

Likewise, the total number of motors produced (m) equals the number of motors sent to the battery division ($0.01b$) plus the number of motors used within the motor division ($0.04m$) plus the number of motors required for sales outside the company (100). This gives the equation $m = 0.01b + 0.04m + 100$.

Thus, the system of two equations in two unknowns (unsimplified) looks like

$$b = 0.03b + 0.08m + 400$$
$$m = 0.01b + 0.04m + 100.$$

You know how to solve this system algebraically by combining terms and using linear combinations or some other technique. However, there is an easier way to do it that uses matrices and takes advantage of technology (calculators or computers) to do the work.

Explore, first, how this system of equations could be represented using matrices. A consumption matrix C has already been defined. If you let

$$P = \begin{bmatrix} b \\ m \end{bmatrix}$$

represent a total production matrix (P) and

$$D = \begin{bmatrix} 400 \\ 100 \end{bmatrix}$$

represent a demand matrix (D), then the system of equations can be described with a matrix equation:

$$\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 0.03 & 0.08 \\ 0.01 & 0.04 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} + \begin{bmatrix} 400 \\ 100 \end{bmatrix}.$$

Perform the matrix multiplication shown here to convince yourself that the matrix equation does, indeed, represent the two algebraic equations previously shown.

This matrix equation can be written more simply as

$$P = CP + D,$$

showing that the total production = internal consumption + external demand.

Notice that this matrix equation resembles the simple linear equation solved in Lesson 7.1 (page 347). Indeed, solving this matrix equation uses the same operations that are used in solving a linear equation. Verify that this is true by tracing the steps for solving a linear equation using ordinary algebra and solving a matrix equation using matrix algebra in the following table.

One-Sector Economy Linear Equation Ordinary Algebra	Two-Sector Economy Matrix Equation Matrix Algebra	Comments
$p = 0.03p + d$	$P = CP + D$	
$p - 0.03p = d$	$P - CP = D$	
$1p - 0.03p = d$	$IP - CP = D$	The identity matrix I times $P = P$.
$(1 - 0.03)p = d$	$(I - C)P = D$	
$\frac{1}{1 - 0.03}(1 - 0.03)p = \frac{1}{1 - 0.03}d$	$(I - C)^{-1}(I - C)P = (I - C)^{-1}D$	*
$1p = \frac{1}{1 - 0.03}d$	$IP = (I - C)^{-1}D$	Recall that $A^{-1}A = I$. (See Exercise 7 in Lesson 3.3, page 127.)
$p = \frac{1}{1 - 0.03}d$	$P = (I - C)^{-1}D$	

* Up to this point the ordinary algebra operations and the matrix operations have been identical. In solving the linear equation, it would be natural to divide both sides of the equation by (1 − 0.03). But, there is no division operation in matrix algebra. The thing to do, then, is to multiply both sides of the linear equation by the multiplicative inverse of (1 − 0.03). Multiplying both sides of the matrix equation by the multiplicative inverse of $(I - C)$ is a valid matrix operation.

In summary, if the consumption matrix (C) and the external demand matrix (D) are known, then the total production matrix (P) can be found using the matrix equation

$$P = (I - C)^{-1}D.$$

For the battery and motor problem, the solution is

$$\begin{bmatrix} b \\ m \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.03 & 0.08 \\ 0.01 & 0.04 \end{bmatrix} \right)^{-1} \begin{bmatrix} 400 \\ 100 \end{bmatrix}.$$

Now, using a calculator or computer to do the computations and rounding to the nearest whole number, you will find that

$$\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 421 \\ 109 \end{bmatrix}.$$

The results show that to fill an order for 400 batteries and 100 motors, the company must produce 421 batteries and 109 motors.

Solving Systems of Linear Equations Using Matrices

The matrix techniques used for solving systems of equations in this lesson can be used to solve any system of n independent equations in n unknowns. Look, for example, at the following system of two equations in two unknowns.

$$2x_1 + 3x_2 = 23$$
$$5x_1 - 2x_2 = 10.$$

This system can be written as a single matrix equation:

$$\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 23 \\ 10 \end{bmatrix}.$$

If we let

$$A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 23 \\ 10 \end{bmatrix},$$

the matrix equation can be written in the form $AX = B$, which is similar to a simple linear equation such as $ax = b$.

One way to solve this linear equation is to multiply both sides of the equation by the multiplicative inverse of a, $\frac{1}{a}$. The same strategy can be used to solve the matrix equation as shown in the following table.

	Linear Equations Ordinary Algebra	Matrix Equations Matrix Algebra
Step 1	$ax = b$	$AX = B$
Step 2	$\frac{1}{a}ax = \frac{1}{a}b$	$A^{-1}AX = A^{-1}B$
Step 3	$1x = \frac{1}{a}b$	$IX = A^{-1}B$
Step 4	$x = \frac{1}{a}b$	$X = A^{-1}B$

Applying this method to the previous system of equations, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 23 \\ 10 \end{bmatrix}.$$

Use a calculator or computer to do the calculations and verify that the solution for the given system of linear equations is $x_1 = 4$ and $x_2 = 5$.

Exercises

Use either a calculator or computer software to perform matrix operations in the following exercises.

1. The total production for the high-tech company described in Exercises 2 and 3 in Lesson 7.1 (pages 350 and 351) over a period of time is \$40,000 worth of computer chips and \$50,000 worth of computers.
   a. Write a consumption matrix, C, for this company. Label the rows and columns of your matrix.
   b. Write a production matrix, P, and label the rows and columns.
   c. Compute the matrix product CP to find the amount of each product that the company uses internally.
   d. Use the information from parts b and c and the matrix equation $D = P - CP$ to compute the amount of computer chips and computers available for sales outside the company (external demand).
   e. The company has a order for \$20,000 worth of computer chips and \$70,000 worth of computers. Find the total production of computer chips and computers necessary to fill this order. Use the matrix equation $P = (I - C)^{-1}D$.

2. a. Use matrices to compute the results for parts c, d, and e of Exercise 4 in Lesson 7.1 (pages 351 and 352).
   b. The company must meet external demands of \$25 million in service and \$50 million in products over a period of time. What must the total production in service and products to meet this demand?

3. Use matrices to compute the results for parts c and g of Exercise 5 in Lesson 7.1 (pages 352 and 353).

4. The techniques developed in this lesson using a two-sector economy can easily be extended to solve problems that involve economies of more than two sectors. For example, look at an economy that has three sectors—transportation, energy, and manufacturing. Each of these

sectors uses some of its own products or services as well as some from each of the other sectors, as follows:

Transportation Sector: Every dollar's worth of transportation provided requires an input of 10 cents' worth of transportation services, 15 cents' worth of energy, and 25 cents' worth of manufactured goods.

Energy Sector: Every dollar's worth of energy produced requires an input of 25 cents' worth of transportation services, 10 cents' worth of energy, and 20 cents' worth of manufactured goods.

Manufacturing Sector: Every dollar's worth of manufactured goods produced requires an input of 20 cents' worth of transportation services, 20 cents' worth of energy, and 15 cents' worth of manufactured goods.

a. Draw a weighted digraph for this three-sector economy.
b. Construct a consumption matrix (C) for this economy. Label the rows and columns of your matrix.
c. The total production over a period of time for this economy is \$150 million in transportation, \$200 million in energy, and \$160 million in manufactured goods. Write a production matrix (P) for this economy. Label the rows and columns of your matrix.
d. Compute the matrix product CP to find the amount of each product that is used internally by the economy. Write your answer as a matrix and label the rows and columns.
e. Use the information from parts c and d and the matrix equation $D = P - CP$ to find the amount of goods available for external demand (sales outside the three sectors described here).
f. The estimated consumer demand for transportation, energy, and manufactured goods and services in millions of dollars are 100, 95, and 110, respectively. Find the total production necessary to fulfill these demands. Use the matrix equation $P = (I - C)^{-1}D$.

5. An economy consisting of three sectors (services, manufacturing, and agriculture) has the consumption matrix

$$C = \begin{array}{r} \text{Services} \\ \text{Manufacturing} \\ \text{Agriculture} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Services} & \text{Manufacturing} & \text{Agriculture} \end{array} \\ \left[\begin{array}{ccc} 0.1 & 0.3 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.2 \end{array} \right] \end{array}$$

a. Draw a weighted digraph for this economy.
b. On which sector of the economy is manufacturing the most dependent? The least dependent?
c. If the services sector has an output of \$40 million dollars, what is the input in dollars from manufacturing? From agriculture?
d. A production matrix, P, in millions of dollars for this economy is as follows. Use the matrix product CP to find the internal consumption of services and products within this economy. Find the external demand matrix D.

$$P = \begin{matrix} \text{Services} \\ \text{Manufacturing} \\ \text{Agriculture} \end{matrix} \begin{bmatrix} 20 \\ 25 \\ 15 \end{bmatrix}.$$

e. An external demand matrix, D, in millions of dollars follows. How much must be produced by each sector to meet this demand?

$$D = \begin{matrix} \text{Services} \\ \text{Manufacturing} \\ \text{Agriculture} \end{matrix} \begin{bmatrix} 4.6 \\ 5.0 \\ 4.0 \end{bmatrix}.$$

6. An economy consisting of four sectors (transportation, manufacturing, agriculture, and services) has the consumption matrix (in millions of dollars worth of products)

$$C = \begin{matrix} \\ \text{Transportation} \\ \text{Manufacturing} \\ \text{Agriculture} \\ \text{Services} \end{matrix} \begin{matrix} \text{Trans.} & \text{Manu.} & \text{Agri.} & \text{Serv.} \\ \left[\begin{matrix} 0.25 \\ 0.15 \\ 0.19 \\ 0.20 \end{matrix}\right. & \begin{matrix} 0.28 \\ 0.15 \\ 0.20 \\ 0.24 \end{matrix} & \begin{matrix} 0.22 \\ 0.17 \\ 0.21 \\ 0.19 \end{matrix} & \left.\begin{matrix} 0.20 \\ 0.23 \\ 0.15 \\ 0.25 \end{matrix}\right] \end{matrix}$$

a. Draw a weighted digraph for this economy.
b. On which sector of the economy is services the most dependent? The least dependent?
c. If the manufacturing sector has an output of \$20 million, what is the input in dollars from services? From transportation?
d. A production matrix, P, in millions of dollars, follows. Use the matrix product CP to find the internal consumption of services and products within this economy. Find the external demand matrix D.

$$P = \begin{matrix} \text{Transportation} \\ \text{Manufacturing} \\ \text{Agriculture} \\ \text{Services} \end{matrix} \begin{bmatrix} 50 \\ 40 \\ 45 \\ 50 \end{bmatrix}.$$

e. An external demand matrix, D, in millions of dollars, follows. How much must be produced by each sector to meet this demand?

$$D = \begin{matrix} \text{Transportation} \\ \text{Manufacturing} \\ \text{Agriculture} \\ \text{Services} \end{matrix} \begin{bmatrix} 10 \\ 12 \\ 10 \\ 15 \end{bmatrix}.$$

7. A two-industry system consisting of services and manufacturing has the consumption matrix

$$C = \begin{matrix} \\ \text{Services} \\ \text{Manufacturing} \end{matrix} \begin{matrix} \text{Services} & \text{Manufacturing} \\ \left[\begin{matrix} 0.5 \\ 0.2 \end{matrix}\right. & \left.\begin{matrix} 0.5 \\ 0.3 \end{matrix}\right] \end{matrix}.$$

a. Compute the total production necessary to satisfy a consumer demand for 15 units of services and 25 units of manufacturing.
b. Comment on the productivity of this system. Explain your answer.
c. If the consumer demand is for 30 units of services and 50 units of manufacturing, find the production needed to fill these demands.
d. On the basis of the results in parts a and b above, predict the total production of services and goods for a consumer demand for 45 units of service and 75 units of manufacturing. Check your prediction by computing the production matrix for this case.

8. A company has two divisions: service and production. The flow of goods and services within this company is described by the consumption matrix

$$C = \begin{matrix} \\ \text{Services} \\ \text{Products} \end{matrix} \begin{matrix} \text{Services} & \text{Products} \\ \left[\begin{matrix} 0.10 \\ 0.05 \end{matrix}\right. & \left.\begin{matrix} 0.25 \\ 0.10 \end{matrix}\right] \end{matrix}.$$

a. Draw a weighted digraph for this situation.
b. The total output for the company during 1 year is \$50 million in services and \$75 million in products. How much of the total output is used internally by each of the company's divisions?
c. What total output is needed to meet an external consumer demand of \$15 million dollars in service and \$25 million in products?
d. If the consumer demand increases to \$22 million for services and to \$30 million for products, what will be the effect on the total production of goods and services?

Projects

9. Research and report to the class on the following.

a. What computer software is available in your school for solving systems of equations?

b. What is the largest number of variables that the software can handle?

c. How long does it take the computer to solve a system with the largest number of variables possible using this software?

d. It took Professor Leontief 56 hours to solve a system of 42 equations in 42 unknowns using the Mark II in the 1940s and 3 minutes to solve a system of 81 equations in 81 unknowns using the IBM 7090 in the 1960s. If the software available in your school can handle systems of 42 and 81 equations, find out:

i. How long it takes to solve a system of 42 independent equations in 42 unknowns.

ii. How long it takes to solve a system of 81 independent equations in 81 unknowns.

e. Investigate parts a and b, for computer software such as *Mathematica, Maple,* or *Theorist* that your school may not own.

10. Research and report to the class on parts a to c of Exercise 9 for the graphing calculators that are available in your school. Try, in particular, to find this information for the TI-92 calculator.

11. Research and report to the class about the size, cost, and capability of the Mark II and IBM 7090 computers. In your report, compare the information you find about these computers with similar information about the personal computers that are available to students in your school.

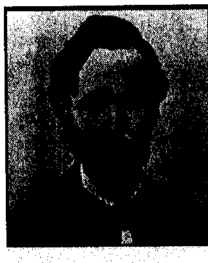
Computer/Calculator Explorations

12. Write a graphing calculator (or computer) program that uses matrices for solving systems of n equations in n unknowns.

13. Write a graphing calculator (or computer program) designed to solve the various consumption problems presented in this lesson.

Markov Chains

A **Markov chain** is a process that arises naturally in problems that involve a finite number of events or states that change over time. In this lesson, a situation that illustrates the significant characteristics of a Markov chain is introduced.



Mathematician of Note

Russian mathematician A. A. Markov's (1856–1922) studies of linked chains of events led to the modern study of stochastic processes. As a result of his work, one type of a stochastic process is called a Markov chain.

Consider the following: Students at Lincoln High have two choices for lunch. They can either eat in the cafeteria or eat elsewhere. The director of food service is concerned about being able to predict how many students can be expected to eat in the cafeteria over the long run. She has asked the discrete mathematics class to help her out by conducting a survey of the student body during the first two weeks of school. The results of the survey show that if a student eats in the cafeteria on a given day, the probability that he or she will eat there again the next day is 70% and the probability that he or she will not eat there is 30%. If a student does not eat in the cafeteria on a given day, the probability that he or she will eat in the cafeteria the next day is 40% and the probability that he or she will not eat there is 60%. On Monday, 75% of the students ate in the cafeteria and 25% did not. What can be expected to happen on Tuesday?

A good way to organize all these statistics is with a tree diagram, similar to the way you organized probabilities in Chapter 6 (see Figure 7.2).

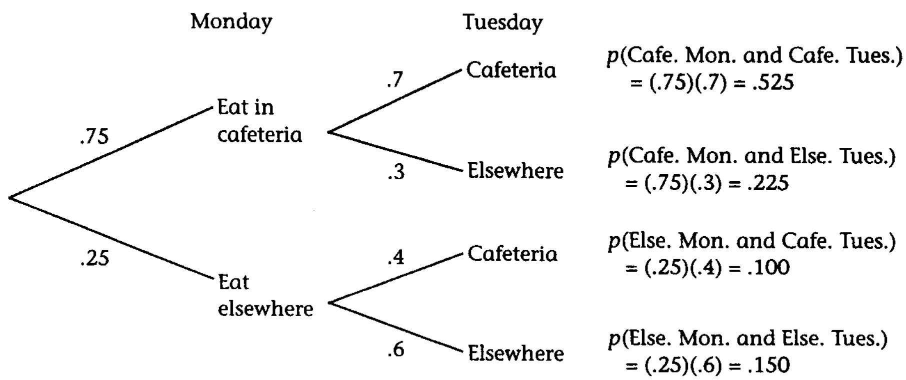


Figure 7.2 Cafeteria statistics organized in a tree diagram.

The director wants to know what portion of the students can be expected to eat in the cafeteria on Tuesday. Look at the tree diagram and notice that this happens if a student eats in the cafeteria on Monday and on Tuesday, or if a student eats elsewhere on Monday and in the cafeteria on Tuesday. This portion is .525 + .100 = .625, or 62.5%. Similarly, the portion of students who will eat elsewhere on Tuesday is .225 + .150 = .375, or 37.5%. Note that this could also be calculated by subtracting .625 from 1.

The tree diagram model is fine if only two stages are required to reach a solution. The director, however, is interested in continuing this process for many days. Because the number of branches of the tree diagram doubles with each additional day, the model soon becomes impractical and so an alternative is needed.

The Monday student data are called the **initial distribution** of the student body and can be represented by a row (or **initial-state**) vector, D_0, where

$$D_0 = \begin{matrix} C & E \\ [.75 & .25] \end{matrix} \qquad \begin{matrix} C = \text{eats in the cafeteria} \\ E = \text{eats elsewhere.} \end{matrix}$$

Movement from one state to another is often called a **transition**, so the data about how students choose to eat from one day to the next is written in a matrix called a **transition matrix**, T, where

$$T = \begin{matrix} & \begin{matrix} C & E \end{matrix} \\ \begin{matrix} C \\ E \end{matrix} & \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} \end{matrix}.$$

Notice that the entries of a transition matrix are probabilities, values between 0 and 1 inclusive. Also notice that the transition matrix is a square matrix and the sum of the probabilities in any row is 1.

Now calculate the product of matrix D_0 and matrix T:

$$D_0T = [.75 \quad .25]\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = [.75(.7) + .25(.4) \quad .75(.3) + .25(.6)]$$

$$= [.625 \quad .375].$$

Compare these calculations with those made in the tree diagram model. The values in the resulting row vector can be interpreted as the portion of students who eat in the cafeteria and who eat elsewhere on Tuesday. This row vector is called D_1 to indicate that it occurs one day after the initial day. To see what happens on Wednesday, it is only necessary to repeat the process using D_1 in place of D_0:

$$D_1T = [.625 \quad .375]\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = [.625(.7) + .375(.4) \quad .625(.3) + .375(.6)]$$

$$= [.5875 \quad .4125].$$

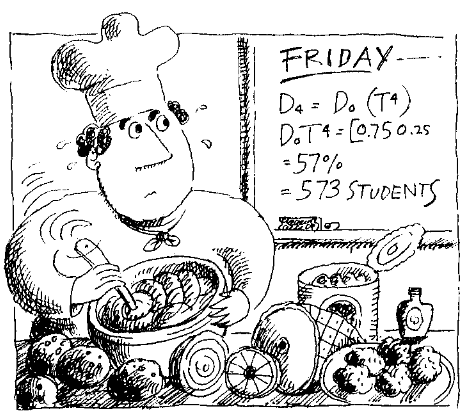
The resulting row vector is called D_2 to indicate that it occurs two days after the initial day. Thereafter D_2 shows that approximately 59% of the students will eat in the cafeteria on Wednesday and 41% will eat elsewhere.

Consider how D_2 was calculated.

$$D_2 = D_1T, \text{ but } D_1 = D_0T, \text{ so by substitution, } D_2 = (D_0T)(T).$$

Because matrix multiplication is associative,

$$D_2 = (D_0T)(T) = D_0(T^2).$$



This means that the calculation of the distribution of students on Wednesday can be completed by taking the initial-state vector times the square of the transition matrix.

This observation simplifies additional calculations. If, for example, you want to know the distribution on Friday, four days from Monday, calculate $D_4 = D_0(T^4)$ on a calculator that has matrix features or on a computer equipped with matrix software:

$$D_0T^4 = [.75 \quad .25]\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}^4 = [.572875 \quad .427125].$$

About 57% of the students can be expected to eat in the cafeteria on Friday. For a school of 1,000 students, about 573 of them can be expected in the cafeteria on that day.

The movement of students from one state to another can also be shown with a weighted digraph called a **transition digraph** or **state diagram** (see Figure 7.3).

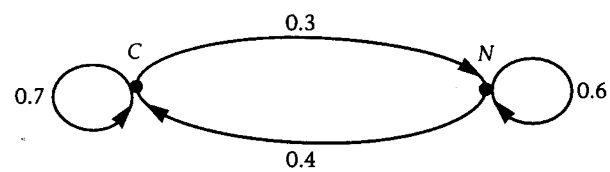


Figure 7.3 Transition digraph for the cafeteria statistics.

Exercises

Use either a calculator or computer software to perform matrix operations in the following exercises.

1. a. Find the distribution of students eating and not eating in the cafeteria each day for the first week of school using the initial distribution D_0 = [0.75 0.25] and the transition matrix T of this lesson.
   b. Find the distribution of students eating and not eating in the cafeteria after 2 weeks (10 school days) have passed. Repeat for 3 weeks (15 days).
   c. What would your report to the director of food services be, based on your computations in parts a and b?
   d. Choose any other initial distribution of students and repeat parts a and b.
   e. Compare the results of parts b and d. Does the initial distribution appear to make a difference in the long run?
   f. Calculate the 15th power of matrix T. Compare the entries in T^{15} to the distribution after the 15th day.

2. When successive applications of a Markov process are made and the rows of powers of the transition matrix converge to a single vector, this common vector is called the **stable-state vector** for the Markov chain. A sufficient condition, which we will not prove in this text, for a Markov chain to have a stable-state vector is that some power of its transition matrix have only positive entries. Since all the entries in the transition matrix T are nonzero probabilities, this condition is clearly met for the cafeteria Markov chain.

a. What is the stable-state vector for the transition matrix T?
b. Make a conjecture about the relationship between the distribution of students in the long run and the stable-state vector of the transition matrix.

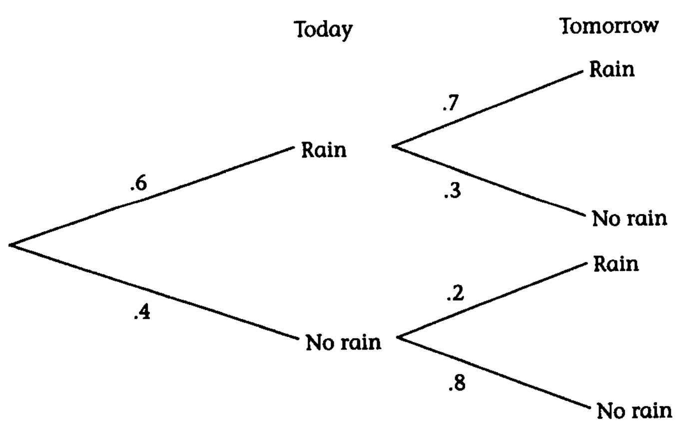
3. Suppose the entire student body eats in the cafeteria on the first day of school. The initial distribution in this case is $D_0 = [1 \quad 0]$. Repeat parts a and b of Exercise 1 for this distribution. After several weeks, what percentage of students will be eating in the cafeteria?

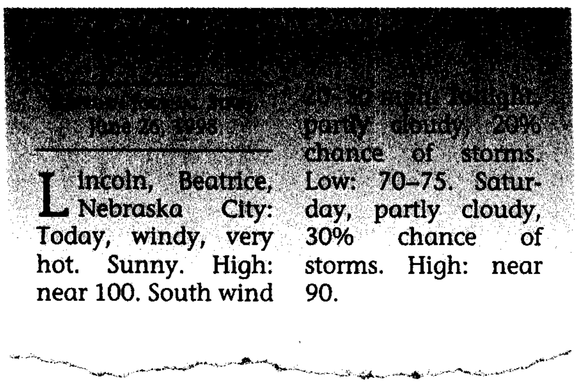
4. Which of the matrices below could be Markov transition matrices? For the matrices that could not be transition matrices, explain why not.

a. $\begin{bmatrix} .7 & .3 \\ .6 & .6 \end{bmatrix}$. b. $\begin{bmatrix} .1 & .4 & .5 \\ .2 & .6 & .2 \end{bmatrix}$. c. $\begin{bmatrix} 1.2 & -4 \\ 1 & 0 \end{bmatrix}$.

d. $\begin{bmatrix} .6 & .3 & .1 \\ .3 & .3 & .3 \end{bmatrix}$. e. $\begin{bmatrix} .75 & .25 \\ 1 & 0 \end{bmatrix}$. f. $\begin{bmatrix} .45 & .55 \\ .33 & .66 \end{bmatrix}$.

5. There is a 60% chance of rain today. It is known that tomorrow's weather depends on today's according to the probabilities shown in the following tree diagram.





partly cloudy, 20% chance of storms.

Lincoln, Beatrice, Nebraska City: Today, windy, very hot. Sunny. High: near 100. South wind

Low: 70–75. Saturday, partly cloudy, 30% chance of storms. High: near 90.

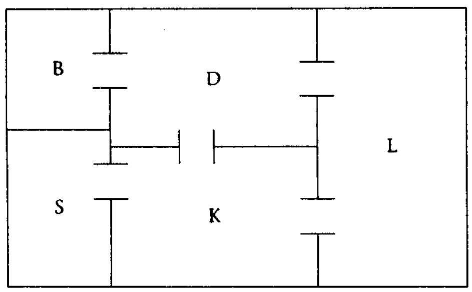
a. What is the probability it will rain tomorrow if it rains today?
b. What is the probability it will rain tomorrow if it doesn't rain today?
c. Write an initial-state matrix that represents the weather forecast for today.
d. Write a transition matrix that represents the transition probabilities shown in the tree diagram.
e. Calculate the forecast for 1 week (7 days) from now.
f. In the long run, for what percentage of days will it rain?

6. A taxi company has divided the city into three districts—Westmarket, Oldmarket, and Eastmarket. By keeping track of pickups and deliveries, the company found that of the fares picked up in the Westmarket district, only 10% are dropped off in that district, 50% are taken to the Oldmarket district, and 40% go to the Eastmarket district. Of the fares picked up in the Oldmarket district, 20% are taken to the Westmarket district, 30% stay in the Oldmarket district, and 50% are dropped off in the Eastmarket district. Of the fares picked up in the Eastmarket district, 30% are delivered to each of the Westmarket and Oldmarket districts, while 40% stay in the Eastmarket district.
a. Draw a transition digraph for this Markov chain.
b. Construct a transition matrix for these data.
c. Write an initial-state matrix for a taxi that starts off by picking up a fare in the Oldmarket district. What is the probability that it will end up in the Oldmarket district after three additional fares?
d. Find and interpret the stable-state vector for this Markov process.

7. Emily, Jon, and Gretchen are tossing a football around. Emily always tosses to Jon, and Jon always tosses to Gretchen, but Gretchen is equally likely to toss the ball to either Emily or Jon.
a. Draw a transition digraph to represent this information.
b. Represent this information as the transition matrix of a Markov chain.
c. What is the probability that Emily will have the ball after three tosses if she was the first one to throw it to one of the others?

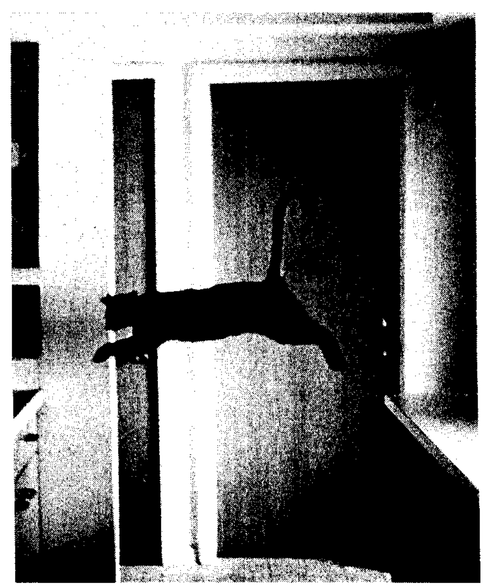
d. Find and interpret the stable-state vector for this Markov chain.
e. Explain why there are no zeros in the stable-state matrix even though there were several zeros in the transition matrix.

8. Jim agreed to care for Emily's cat, Ellington, for the weekend. On Friday night Ellington prowled the first floor of Jim's house, randomly moving from room to room, not staying in one room for more than a few minutes. The following floor plan shows the location of the rooms and doorways in Ellington's range. The letters on the floor plan represent Living room, Dining room, Kitchen, Bathroom, and Study.

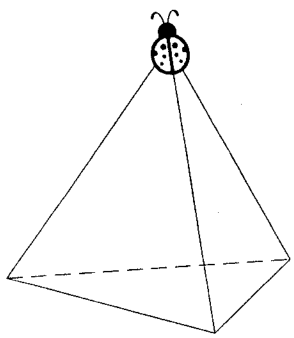


Each of Ellington's movements can be interpreted as a transition in a Markov chain in which a state is identified with the room he is in. The first row of the transition matrix is

$$\begin{array}{cc} & \begin{array}{ccccc} L & D & K & S & B \end{array} \\ L & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}. \end{array}$$



a. Construct the complete transition matrix for this situation.
b. If Ellington starts off in the living room, what is the probability that he will be in the study after two transitions? After three transitions?
c. After a large number of transitions, what is the probability that Ellington will be in the bathroom?
d. In the long run, what percentage of the time will Ellington spend in either the kitchen or the dining room?



9. A discrete mathematics student observes a bug crawling from vertex to vertex along the edges of a tetrahedron model on the teacher's desk (see figure at left). From any vertex the bug is equally likely to go to any other vertex.
   a. Set up a transition matrix for this situation. (Note: To minimize roundoff errors, if your calculator does not accept fractions, approximate 1/3 to at least four decimal places when you enter the data.)
   b. Determine the probabilities for the location of the bug when the passing bell rings if it moves to a different vertex about 20 times during the class period.

10. Dick's old hound dog, Max, spends much of his time during the day running from corner to corner along the fence surrounding his square-shaped yard. There is a .5 probability that Max will turn in either direction at a corner. The corners of Max's yard point north, east, south, and west.
    a. Draw an initial-state diagram that represents Max's movement.
    b. Construct a transition matrix for this situation.
    c. Look at the behavior of successive powers of the transition matrix. Notice the oscillation of the transition probabilities between the states represented by the rows of the matrices. Does this system appear to stabilize in some way? Explain your answer.
    d. Approximately what percentage of the time will Max spend at each of the corners of his yard? (Note: You need to halve the entries in the matrix to account for the oscillating pattern.)
    e. Max changes his routine one day, and the pattern of his new movements is represented by the following transition matrix. Answer part d for this situation.

$$\begin{array}{c} \\ N \\ E \\ S \\ W \end{array}\begin{array}{c} \begin{array}{cccc} N & E & S & W \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{8} & 0 & \frac{5}{8} & 0 \\ 0 & \frac{3}{8} & 0 & \frac{5}{8} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{bmatrix} \end{array}$$

11. Using mathematical induction, prove that $D_k = D_0T^k$ for any original distribution D_0 and transition matrix T, where k is a natural number.

12. A group of researchers are studying the effect of a potent flu vaccine in healthy (well) and infirm (ill) rats. When the rats are injected with the vaccine, three things may occur. The rat may have no reaction where its health status does not change. The rat may have a mild reaction and become ill, or the rat may have a severe reaction and die. The probabilities of each of these reactions are shown in the following matrix.

$$\begin{array}{l} \\ \text{Well} \\ \text{Ill} \\ \text{Dead} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Well} & \text{Ill} & \text{Dead} \end{array} \\ \begin{bmatrix} .8 & .2 & 0 \\ .1 & .6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

a. Write an initial-state vector for a healthy rat who is injected with the vaccine.
b. In this study the scientists check the status of the rats on a daily basis. Use the transition matrix to predict the health of the rat in part a after 4 days.
c. Use the transition matrix to predict the rat's health in the long run.
d. This Markov chain has a state that is called an **absorbing state**. Which state do you think it is? Why?

13. A hospital categorizes its patients as well (in which case they are discharged), good, critical, and deceased. Data show that the hospital's patients move from one category to another according to the probabilities shown in this transition matrix:

$$\begin{array}{l} \\ \text{Well} \\ \text{Good} \\ \text{Critical} \\ \text{Dead} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Well} & \text{Good} & \text{Critical} & \text{Dead} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .5 & .3 & .2 & 0 \\ 0 & .3 & .6 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

a. Write an initial-state matrix for a patient who enters the hospital in critical condition.
b. If patients are reclassified daily, predict the patient's future after 1 week in the hospital.
c. Predict the future of any patient in the long run.
d. Does this Markov chain have any absorbing states? Which states do you think are absorbing? Why?

Project

14. Research and report to the class on the life and work of A. A. Markov.

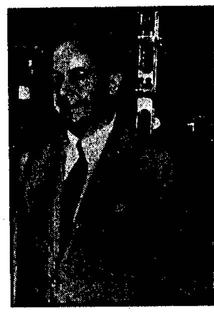
Computer/Calculator Exploration

15. Write a graphing calculator (or computer) program that can be used to find the stable-state vector for a Markov chain.

Lesson 7.4

Game Theory, Part 1

The basic ideas of game theory were first researched by John von Neumann in the 1920s. But, it was not until the 1940s during World War II that game theory was recognized as a legitimate branch of mathematics. Thus, most of the work in this area has been done over the last 60 years.

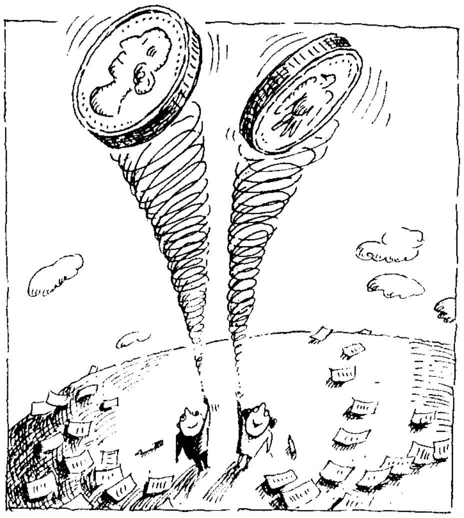


Mathematician of Note

This photograph, taken in 1952, shows John von Neumann (1903–1957) standing in front of the EDVAC computer.

You probably tend to think of games as being fun or relaxing ways to spend your time. There are, however, many decision-making situations in fields such as economics or politics that can also be thought of as games. In such games, there are two or more players who have conflicting interests. These players may be individuals, teams of people, whole countries, or even forces of nature. Each player (or side) has a set of alternative courses of action called **strategies** that can be used in making decisions. Mathematical game theory deals with selecting the best strategies for a player to follow in order to achieve the most favorable outcomes.

In this lesson, you will explore some examples of games with two players and use matrices to determine the best strategy for each player to choose. As the first example, consider a simple coin-matching game that Sol and Tina are playing. Each conceals a penny with either heads or tails turned upward. They display their pennies simultaneously. Sol will win



three pennies from Tina if both are heads. Tina will win two pennies from Sol if both are tails, and one penny from Sol if the coins don't match. What is the best strategy for each player?

If you think carefully about the game, you will probably decide that it isn't such a good deal for Sol. As long as Tina displays tails, she cannot lose. If Sol knows that Tina is going to play tails, he should display heads because he will lose more if he doesn't.

You probably think this is a rather boring game. In a sense it is, because both players will do the same thing every time. A game in which the best strategy for both players is to pursue the same strategy every time is called **strictly determined.** Although strictly determined games are fairly boring, there are situations in life in which they cannot be avoided and knowing how to analyze them properly can be beneficial. Although strictly determined games are often very simple, they can be difficult to analyze without an organizational scheme. Matrices offer a way of doing this.

The following matrix presents Sol's view of the game. It is customary to write a game matrix from the viewpoint of the player associated with the matrix rows rather than the player associated with the columns. Such a matrix is called a **payoff matrix**. The entries are the payoffs to Sol for each outcome of the game.

$$\begin{array}{cc} & \text{Tina} \\ & \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \text{Sol} \begin{array}{c} \text{Heads} \\ \text{Tails} \end{array} & \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \end{array}.$$

Game Theo[illegible] a Nobel

New York Times, October 12, 1994

The 1994 Nobel Memorial Prize in Economic Science, a $930,000 award to be divided among three pioneers in the field of game theory, celebrates achievements in building the foundations for analyzing interactions among businesses, nations, and even biological species.

But just as important, the prize awarded to John F. Nash of Princeton University, John C. Harsanyi of the University of California at Berkeley and Reinhard Selton of the University of Bonn acknowledges a sea of changes in economics that has occurred in the last decade.

John von Neumann and Oskar Morgenstern, economists at Princeton, invented the field. Their

continues

This matrix is easy to follow if you are Sol, but the entries are just the opposite if you are Tina. If you find it difficult to think of all the numbers as their opposites, you may find it preferable to write a second matrix from Tina's point of view:

$$\begin{array}{cc} & \text{Tina} \\ & \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \text{Sol} \begin{array}{c} \text{Heads} \\ \text{Tails} \end{array} & \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}. \end{array}$$

Consider the game from Sol's point of view. Sol does not want to lose any more money than necessary, so he analyzes his strategies from the standpoint of his losses. If he displays heads, the worst he can do is to lose 1 cent. If he displays tails, the worst he could do is lose 2 cents. Since it is better to lose 1 cent than lose 2 cents, Sol decides to display heads.

Sol's analysis can be related to the payoff matrix by writing the worst possible outcome of each strategy to the right of the row that represents it. The worst possible outcome of each strategy is the smallest value of each row, often referred to as the **row minimum**. Sol's best strategy is to select the option that produces the largest of these minimums or, in other words, to select the "best of the worst." Because this value is the largest of the smallest row values, it is called the **maximin** (the maximum of the row minimums).

Game Theory Captures a Nobel (*continued*)

book, published in 1944, "The Theory of Games and Economic Behavior," was the first to delve deeply into the likely consequences of strategic interactions, where all the actors must consider the potential for reaction.

The great bulk of the work by economists in game theory has been in an area where its insights had been most sorely missed: the organization of industry.

An example: Intel, the microprocessor giant, gave up an effective monopoly on the 86-series chip by allowing Advanced Micro Devices to share the technology. Intel, it seems, decided that computer makers would not lock themselves into a new microprocessor technology unless they were protected from future price-gouging by a monopolist. So by licensing another manufacturer, Intel successfully increased the demand for its own product.

Here, game theory explained corporate behavior that made no sense in nonstrategic terms.

Game theorists have also been hired to create corporate strategy from scratch, most notably in the case of the Federal Communications Commission's auction in December 1994 of bands on the radio spectrum for use in wireless communications. Every major bidder hired academic game theorists as consultants. The F.C.C.'s goal was to raise the maximum amount of money, at least $10 billion and it used game theory in attempting to reach that target.

		Tina Heads	Tina Tails	Row minimums
Sol	Heads	3	−1	(1)
	Tails	−1	−2	−2

In general, the best strategy for the row player in a strictly determined game is to select the strategy associated with the largest of the row minimums.

Because Tina's point of view is exactly the opposite of Sol's, she views the minimums as maximums and vice-versa. Therefore, her best strategy is the one associated with the smallest of the largest column values, the **minimax** (the minimum of the column maximums).

		Tina Heads	Tina Tails
Sol	Heads	3	−1
	Tails	−1	−2
Column maximums		3	(1)

In general, the best strategy for the column player in a strictly determined game is to select the strategy associated with the smallest of the column maximums.

Remember, if you find it confusing to reverse your thinking when analyzing the columns, change the sign of all matrix entries and use the same reasoning you used for the rows.

In this game, the value selected by both Sol and Tina is the same one, that is, the -1 that appears in the upper right-hand corner of the matrix. This is the identifying characteristic of strictly determined games. If the value selected by the two players is not the same, then the game is not strictly determined and is much less boring. Games that are not strictly determined are considered in the next lesson.

A strictly determined game is one in which the maximin (the maximum of the row minimums) and the minimax (the minimum of the column

maximums) are the same value. This value is called the saddle point of the game. The saddle point can be interpreted as the amount won per play by the row player.

When players have more than two strategies, a game is somewhat harder to analyze. It is often helpful to eliminate strategies that are **dominated** by other strategies. For example, in a competition between two pizza restaurants, Dino's and Sal's, both are considering four strategies: running no special, offering a free minipizza with the purchase of a large pizza, offering a free medium pizza with the purchase of a large one, and offering a free drink with any pizza purchase.

A market study estimates the gain in dollars per week to Dino's over Sal's according to the following matrix.

		Sal's			
		No special	Mini	Medium	Drink
Dino's	No special	200	−400	−300	−600
	Mini	500	100	200	600
	Medium	400	−100	−200	−300
	Drink	300	0	400	−200

What should the managers of Dino's and Sal's do?

Suppose you are the manager of Dino's and examine the first two rows carefully. You notice that no matter what your competitor does, you always achieve a larger payoff by offering the free mini. It would make no sense, therefore, to run no special. The first row of the matrix is dominated by the second and can be eliminated by drawing a line through it. Similarly, the second row dominates the third, and so the third row can be eliminated.

		Sal's			
		No special	Mini	Medium	Drink
Dino's	No special	~~200~~	~~−400~~	~~−300~~	~~−600~~
	Mini	500	100	200	600
	Medium	~~400~~	~~−100~~	~~−200~~	~~−300~~
	Drink	300	0	400	−200

Now think of the matrix from the point of view of Sal's manager. Because all the payoffs to Sal's are opposites of the payoffs to Dino's, a column is dominated if all its entries are larger, rather than smaller, than those of another column. Notice that all the values in the first column are larger than the corresponding values in the second column. Because the first column is dominated by the second, it is unwise for Sal's to run no special, and so this strategy can be eliminated. Similarly, the second column dominates the third, and so the third column can be eliminated.

		Sal's			
		No special	Mini	Medium	Drink
Dino's	No special	~~200~~	~~−400~~	~~−300~~	~~−600~~
	Mini	~~500~~	100	~~200~~	600
	Medium	~~400~~	~~−100~~	~~−200~~	~~−300~~
	Drink	~~300~~	0	~~400~~	−200

Once these strategies are eliminated, the game is easier to examine for a minimax and a maximin:

		Sal's				
		No special	Mini	Medium	Drink	Row minimums
Dino's	No special	~~200~~	~~−400~~	~~−300~~	~~−600~~	
	Mini	~~500~~	100	~~200~~	600	(100)
	Medium	~~400~~	~~−100~~	~~−200~~	~~−300~~	
	Drink	~~300~~	0	~~400~~	−200	−200
Column maximums			(100)		600	

The game is strictly determined with a saddle point of 100. Dino's best strategy is to offer the free mini, and Sal's best strategy is to do the same. By pursuing this strategy, Dino's will gain about $100 a week over Sal's.

Exercises

1. Each of the following matrices represents a payoff matrix for a game. Determine the best strategies for the row and column players. If the game is strictly determined, find the saddle point of the game.

a. $\begin{bmatrix} 16 & 8 \\ 12 & 4 \end{bmatrix}$. b. $\begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}$. c. $\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$.

d. $\begin{bmatrix} 0 & 1 & 2 \\ 3 & -2 & 0 \end{bmatrix}$. e. $\begin{bmatrix} 0 & -6 & 1 \\ -4 & 8 & 2 \\ 6 & 5 & 4 \end{bmatrix}$. f. $\begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & 2 \\ -1 & -4 & 0 \end{bmatrix}$.

2. a. For the game defined by the following matrix, determine the best strategies for the row and column players and the saddle point of the game.

$$\begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$$

b. Add 4 to each element in the matrix given in part a. How does this affect the best strategies and the saddle point of the game?

c. Multiply each element in the matrix in part a by 2. How does this affect the saddle point of the game and the best strategies?

d. Make a conjecture based on the results of parts b and c.

3. Discuss what would happen in the game given in this lesson if Sol decided to depart from his best strategy. Suppose he switches to displaying tails occasionally. Do you think Tina should still play tails every time? Explain your answer.

4. Use the concept of dominance to solve each of the following games. Give the best row and column strategies and the saddle point of each game.

a. $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{matrix} E & F & G \end{matrix} \\ \begin{bmatrix} 3 & 1 & 7 \\ 0 & 1 & 3 \\ 4 & 3 & 4 \\ 1 & 3 & 6 \end{bmatrix} \end{array}$. b. $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{matrix} E & F & G \end{matrix} \\ \begin{bmatrix} 4 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 5 \\ 3 & 2 & 4 \end{bmatrix} \end{array}$.

5. The Democrats and Republicans are engaged in a political campaign for mayor in a small midwestern community. Both parties are planning their strategies for winning votes for their candidate in the final days. The Democrats have settled on two strategies, A and B, and the Republicans plan to counter with strategies C and D. A local newspaper got wind of their plans and conducted a survey of eligible voters. The results of the survey show that if the Democrats choose plan A and the Republicans choose plan C, then the Democrats will gain 150 votes. If the Democrats choose A and the Republicans choose D, the Democrats will lose 50 votes. If the Democrats choose B and the Republicans choose