C, the Democrats will gain 200 votes. If the Democrats choose B and the Republicans choose D, the Democrats will lose 75 votes. Write this information as a matrix game. Find the best strategies and the saddle point of the game.

6. Two major discount companies, Salemart and Bestdeal, are planning to locate stores in Nebraska. If Salemart locates in city A and Bestdeal in city B, then Salemart can expect an annual profit of $50,000 more than Bestdeal's annual profit. If both locate in city A, they expect equal profits. If Salemart locates in city B and Bestdeal in city A, then Bestdeal's profits will exceed Salemart's by $25,000. If both companies locate in city B, then Salemart's profits will exceed Bestdeal's by $10,000. What are the best strategies in this situation and what is the saddle point of the game?

7. Jon and Gretchen each have three dimes. They both hold either one, two, or three coins in a clenched fist and open their fists together. If they both are holding the same number of coins, Jon will take the coins that Gretchen is holding. If they are holding different numbers of coins, then Gretchen will take the coins that Jon is holding.
   a. Write the payoff matrix from Jon's point of view.
   b. Does this game have a saddle point? If so, what are the best strategies for Jon and Gretchen?

8. Mike is going over to see his girlfriend, Nancy, after track practice, when he suddenly remembers that today may be a special anniversary for Nancy and him, and he always brings her a single red rose on this occasion. But he's not sure. Maybe the anniversary is next week. What should he do? If it is their anniversary and he doesn't bring a rose, then he'll be in bad trouble. On a scale from 0 to 10, he'd score a -10. If he doesn't bring a rose and it isn't their anniversary, Nancy won't know anything about his frustration and he'll score a 0. If he brings a rose and it is not their anniversary, then Nancy will be suspicious that something funny is going on but he'll score about a 2. If it is their special anniversary and he brings a rose, then Nancy will be expecting it and he'll score a 5. Write a payoff matrix for this situation. What is Mike's best strategy?

9. School board and Teacher Education Association representatives are meeting to negotiate a contract. Each side can either threaten (reduction in staff or strike), refuse to negotiate, or negotiate willingly. Each side decides its strategy prior to coming to the negotiating table. The

following payoff matrix gives the percentage pay increases for the teachers that would result from each combination of strategies. Find the best strategies for each side.

		School Board		
		Threaten	Refuse	Negotiate
Teachers	Threaten	5	4	3
	Refuse	3	0	2
	Negotiate	4	3	2

Projects

10. Research and report to the class on the life and work of John von Neumann.

11. Research and report to the class on the development of game theory during World War II.

12. Find and report to the class on applications of game theory in foreign policy, political science, economics, or business.

Lesson 7.5

Game Theory, Part 2

The games considered in the previous lesson were strictly determined. In this lesson, games in which there is not a single best strategy for each player are introduced.

Look again at the game of the previous lesson. Suppose that Sol, knowing that he can lose only if Tina plays rationally, proposes changing the game. He now will win four pennies if both coins are heads and one penny if both coins are tails. He will lose two pennies if he shows heads and Tina shows tails, and three pennies if he shows tails and Tina shows heads. The new payoff matrix is:

$$\begin{array}{ccc} & & \text{Tina} \\ & & \begin{array}{cc}\text{Heads} & \text{Tails}\end{array} \\ \text{Sol} & \begin{array}{c}\text{Heads}\\ \text{Tails}\end{array} & \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \end{array}.$$

Here is the same matrix with the row minimums and column maximums:

		Tina: Heads	Tina: Tails	Row minimums
Sol	Heads	4	−2	(−2)
	Tails	−3	1	−3
Column maximums		4	(1)	

The maximin is −2 and the minimax is 1. Since the maximin does not agree with the minimax, the game is not strictly determined. The best strategy for either player is to display a mixture of heads and tails and keep the other player guessing. One way to do this would be to flip the coin and allow it to appear heads or tails at random. But such a strategy would cause heads and tails to appear in roughly equal portions, and it is not clear that this would be best for either player. Another strategy Sol could try is to roll a die and show heads if one, two, three, or four spots appeared, and tails otherwise. He might reason that this would benefit him because he would show heads two-thirds of the time and he wins the most if two heads appear.

Consider what will happen if Sol and Tina each decide to flip their coins. The probability of heads is .5, as is the probability of tails. Because Sol's flip and Tina's flip are made independently, the probability of both showing heads or both showing tails is $.5 \times .5 = .25$. The same is true for the cases in which one shows a head and the other shows a tail (see Figure 7.4).

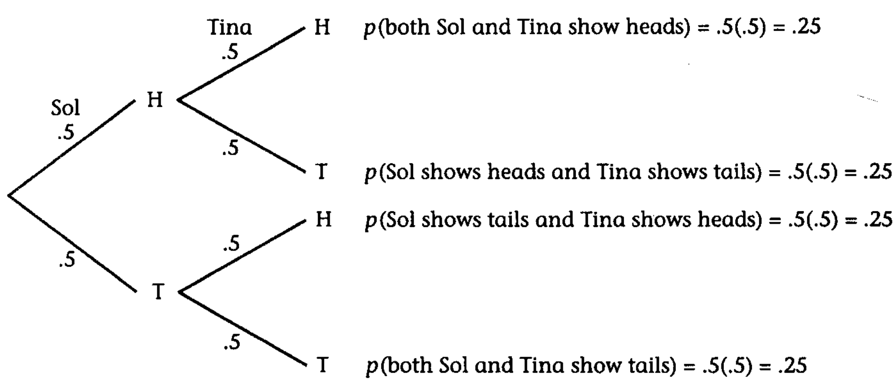


Figure 7.4 Probabilities when Sol and Tina flip their coins.

The probability distribution for Sol's winnings for this case are shown in the following table.

Outcome	HH	HT	TH	TT
Probability	.25	.25	.25	.25
Amount won	4	−2	−3	1

The expected payoff of the game for Sol is .25(4) + .25(−2) + .25(−3) + .25(1) = 1.00 − .50 − .75 + .25 = 0.

Since Tina's payoffs are the opposite of Sol's, her expectation is .25(−4) + .25(2) + .25(3) + .25(−1) = −1.00 + .50 + .75 − .25 = 0. If both players display heads and tails in equal proportions in this way, the game is **fair** because their expectations are equal.

But suppose that Tina decides to play heads 40% of the time, while Sol continues flipping his coin. The probability of both heads is now .5 × .4 = .2, while the probability of both tails is .5 × .6 = .3. The probability that Sol shows heads and Tina shows tails is .5 × .6 = .3 and that Sol shows tails and Tina shows heads is .5 × .4 = .2 (see Figure 7.5).

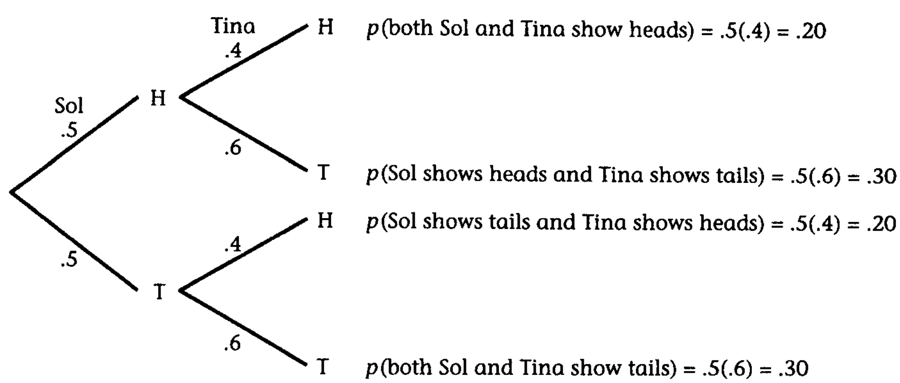


Figure 7.5 Probabilities when Sol flips his coin and Tina shows heads 40% of the time.

The distribution for Sol's winnings now looks like this:

Outcome	HH	HT	TH	TT
Probability	.2	.3	.2	.3
Amount won	4	−2	−3	1

The expected payoff for Sol is now .2(4) + .3(−2) + .2(−3) + .3(1) = .8 − .6 − .6 + .3 = −.1. This means he will lose 0.1 pennies per play, or 1 penny every 10 plays. Tina has an advantage and the game is no longer fair!

You have seen that Tina can gain an advantage over Sol if she knows he will display heads and tails in equal proportions. She does not know that Sol is going to do this, however, so how can she decide her best mixture of strategies? How can Sol decide what is best for him?

Reconsider the game from Sol's point of view, and suppose that Tina plays heads every time while Sol continues to flip his coin. The outcomes for this combination is shown in Figure 7.6. Sol's expected payoff is now .5(4) + .5(−3) = .20 − 1.5 = −1.3.

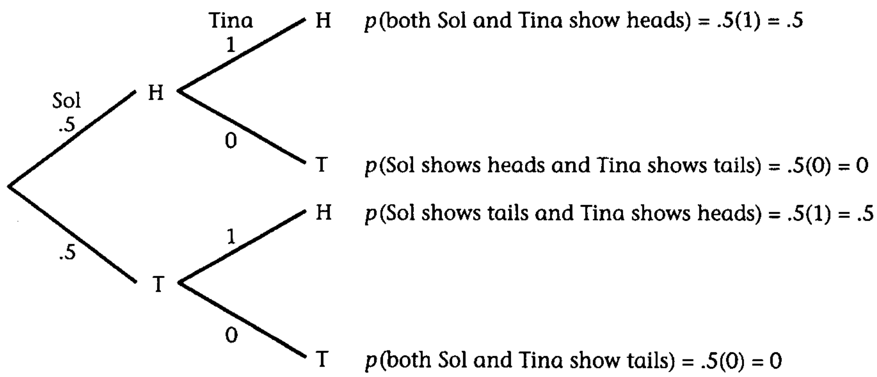


Figure 7.6 Probabilities when Sol flips his coin and Tina always plays heads.

If Tina decides to play tails each time while Sol continues to flip his coin (see Figure 7.7), Sol's expectation is .5(−2) + .5(1) = −1.0 + .5 = −.5.

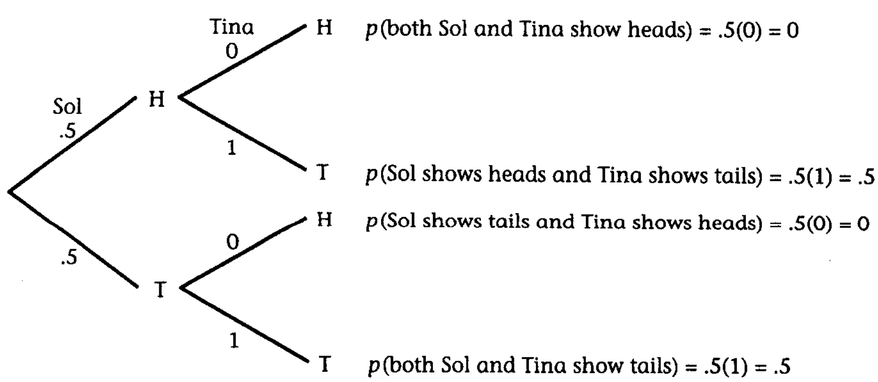


Figure 7.7 Probabilities when Sol flips his coin and Tina always plays tails.

Another way to show these calculations is to write the probabilities of Sol's displaying heads and tails in a row matrix and find the matrix product:

$$[.5 \quad .5]\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = [.5(4) + .5(-3) \quad .5(-2) + .5(1)]$$
$$= [.2 - 1.5 \quad -1 + .5] = [-1.3 \quad -.5].$$

Suppose Sol switches to displaying heads 60% of the time. Then this matrix product is

$$[.6 \quad .4]\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = [.6(4) + .4(-3) \quad .6(-2) + .4(1)]$$
$$= [2.4 - 1.2 \quad -1.2 + .4] = [1.2 \quad -.8].$$

This means that if Sol displays heads 60% of the time, he will gain 1.2 pennies per play if Tina always displays heads and lose 0.8 pennies per play if Tina always displays tails.

In general, if the probability Sol will display heads is p, his expected winnings per play, if Tina displays all heads or all tails, are

$$[p \quad 1-p]\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = [4p - 3(1-p) \quad -2p + 1(1-p)].$$

Because it is not very likely that Tina will display all heads or all tails, Sol's best strategy is to act in such a way that the two expectations are balanced or equalized. To find the value of p that does this, set the two expectations equal to each other and solve the resulting equation.

$$\begin{aligned} 4p - 3(1-p) &= -2p + 1(1-p) \\ 4p - 3 + 3p &= -2p + 1 - p \\ 7p - 3 &= -3p + 1 \\ 10p &= 4 \\ p &= .4 \\ 1 - p &= .6. \end{aligned}$$

Sol's best strategy is to display heads four-tenths of the time and tails six-tenths of the time. One way he could accomplish this is to generate a random number on a calculator and display heads if the number that comes up is less than or equal to 0.4.

Tina's best strategy can be determined in a similar way. Call the probability that she displays heads q. Because she is the column player, multiply the payoff matrix times a column matrix to obtain her expectations if Sol plays either all heads or all tails:

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} 4q-2+2q \\ -3q+1-q \end{bmatrix}.$$

Equate the two entries in the resulting matrix and solve to find Tina's best strategy.

$$\begin{aligned} 4q-2+2q &= -3q+1-q \\ 6q-2 &= -4q+1 \\ 10q &= 3 \\ q &= .3 \\ 1-q &= .7. \end{aligned}$$

In this case, Tina's best strategy is to display heads three-tenths of the time and tails seven-tenths of the time.

If both players pursue these strategies, the probability that a pair of heads will appear is .4(.3) = .12, or 12% of the time, and that a pair of tails will appear is .6(.7) = .42, or 42% of the time. The probability that Sol shows heads and Tina shows tails is .4(.7) = .28 or 28% of the time, and that Sol shows tails and Tina shows heads is .6(.3) = .18, or 18% of the time (see Figure 7.8).

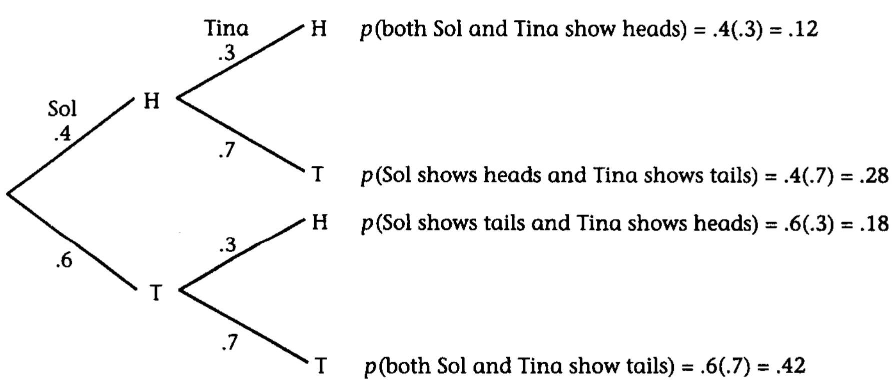


Figure 7.8 Probabilities when both Sol and Tina play their best strategies.

The resulting probability distribution from Sol's point of view is:

Outcome	HH	HT	TH	TT
Probability	.12	.28	.18	.42
Amount won	4	−2	−3	1

Sol's expected payoff for the game is $.12(4) + .28(-2) + .18(-3) + .42(1) = .48 - .56 - .54 + .42 = -.2$. If both players pursue their best strategy, the game is in Tina's favor and she can expect to win 0.2 of a penny per play, or 2 pennies every 10 plays, from Sol.

Using Matrices to Find the Expected Payoff

The expected payoff for a game can be found very easily by using matrices and a calculator. To do this, we form a row matrix A using the probabilities for the row player and a column matrix C using the probabilities of the column player. Then the expected payoff for the game for the row player equals the matrix product ABC, where B is the payoff matrix for the game.

In the preceding example, in which both Sol and Tina play their best strategies,

$$A = [.4 \quad .6],\ C = \begin{bmatrix} .3 \\ .7 \end{bmatrix}, \text{ and}$$

$$ABC = [.4 \quad .6]\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} .3 \\ .7 \end{bmatrix} = -.2,$$

as shown on the following calculator screens.

```
[A]
     [[.40 .60]]
[B]
 [[4.00   -2.00]
  [-3.00 1.00 ]]
```

```
[C]
          [[.30]
           [.70]]
[A]*[B]*[C]
         [[-.20]]
```

A Graphical Solution for Sol's Best Strategy

Sol's search for a best strategy can be visualized graphically:

1. Draw a horizontal line to represent the probability of Sol's displaying heads. Scale this axis in tenths from 0 to 1 (see Figure 7.9).
2. Draw vertical axes at each end of the horizontal axis and scale them from the minimum amount Sol can win (—3 in this case) to the maximum amount (4 in this case).
3. Draw a diagonal line to represent what happens if Tina always displays heads. To do this, notice that if Tina displays heads and Sol displays tails, Sol will lose 3 cents. Place a dot at −3 on the vertical axis on the left, where the probability of Sol's displaying heads is 0. Similarly, if Sol displays heads and Tina displays heads, he will win 4 cents. Place a dot at the 4 on the vertical axis on the right where the probability of Sol's displaying heads is 1. Connect the two dots with a diagonal line. Sol's expected winnings for his various strategies for displaying heads when Tina always displays heads can be read from this line (see line 1 in Figure 7.9).
4. Repeat the procedure in step 3 to draw a diagonal line that shows what happens if Tina always displays tails. Place a dot at 1 on the

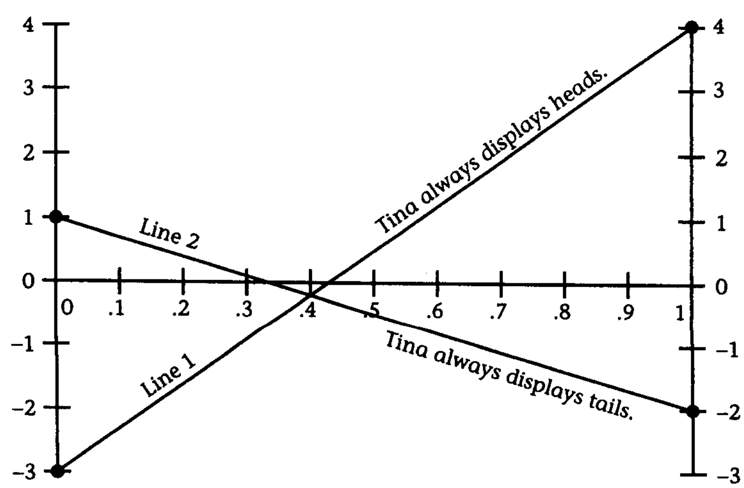


Figure 7.9 Sol's best strategy for displaying heads.

left vertical axis since Sol wins 1 cent when both he and Tina show tails. Similarly, place a dot at -2 on the vertical axis on the right since Sol loses 2 cents if he displays heads and Tina displays tails. Connect the two dots. Sol's expected winnings for his various strategies when Tina always displays tails can be read from this line (see line 2 in Figure 7.9).

Since Sol's best strategy is to act in such a way that the two expectations are balanced or equalized, the intersection of the two lines lies directly below Sol's best strategy for displaying heads. His expected payoff for playing this strategy can be read from the vertical axes.

To calculate the exact values for Sol's best strategy and his expected payoff from the graph, let x represent the probability of Sol's displaying heads and P represent the payoff. Then the equations of line 1 and line 2 in slope intercept form are $P = 7x - 3$ and $P = -3x + 1$, respectively. Setting these two equations equal and solving for x, we get

$$\begin{aligned} 7x - 3 &= -3x + 1 \\ 10x &= 4 \\ x &= .4 \end{aligned}$$

Substituting .4 for x in the first equation, we get

$$P = 7x - 3 = 7(.4) - 3 = 2.8 - 3 = -.2$$

Exercises

1. Suppose that in the example of this lesson, Sol decides to return to flipping his coin while Tina continues to pursue her best strategy of playing heads three-tenths of the time.

a. Set up a tree diagram to compute the probabilities of each of the four outcomes for this game.
b. What is the probability that both Sol and Tina will show heads?
c. What is the probability that Tina will show tails and Sol will show heads?
d. What is the probability that Tina will show heads and Sol will show tails?
e. What is the probability that both Sol and Tina will show tails?
f. Write a probability distribution chart for Sol's winnings.
g. Calculate Sol's expected payoff for this game. Explain what this means in terms of pennies won or lost.
h. How does this payoff compare with Sol's expectation if he plays his best strategy as computed in this lesson?

2. Use matrices and a calculator as shown on page 388 to find the expected payoffs for Sol in this exercise.

a. Suppose that in the example of this lesson, Sol decides to play heads three-fourth of the time while Tina continues to pursue her best strategy of playing heads three-tenths of the time. Find Sol's expectation for this situation.

b. Choose two or three other strategies for Sol to play while Tina continues to pursue her best strategy of playing heads three-tenths of the time. Compute Sol's expected payoff for these strategies.

c. Suppose now that Sol returns to using his best strategy of playing heads four-tenths of the time while Tina plays a variety of strategies. Choose three or four different strategies for Tina to play while Sol plays his best strategy and find Sol's expected payoff in each case.

d. Compare your results with your classmates' results for Sol's expectation in parts a to c. Make a conjecture based on your observations.

3. Suppose that Sol and Tina change their game so that the payoffs to Sol are

$$\begin{array}{ccc} & & \text{Tina} \\ & & \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \text{Sol} & \begin{array}{c} \text{Heads} \\ \text{Tails} \end{array} & \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}. \end{array}$$

a. Use the row matrix $[p \quad 1-p]$ to find Sol's best strategy for this game.

b. Use the column matrix $\begin{bmatrix} q \\ 1-q \end{bmatrix}$ to find Tina's best strategy for this game.

c. Set up a tree diagram to compute the probabilities of each of the four outcomes for this game.

d. Prepare a probability distribution chart for Sol's winnings.

e. Find Sol's expectation for this game.

f. Interpret your answer in part e in terms of how many pennies Sol can expect to win or lose over a number of games.

g. Construct a graph showing Sol's best strategy for playing heads in this game.

h. Find the equations of the lines in part g. Set these equations equal and find Sol's best strategy and his expected payoff for this game.

4. The procedure outlined in this lesson is designed to determine the best mixture of strategies when a game is not strictly determined. Therefore, you should always inspect a game to see whether it is strictly determined and apply the saddle point technique of the previous lesson if it is. It is, however, easy to forget to do this. To see what will happen if you attempt to determine a mixture of strategies for a game that is strictly determined, apply the techniques of this lesson to the strictly determined game that Sol and Tina were playing in the last lesson and try to find the best mixture of strategies for each of them. The payoff matrix for this game is reprinted here.

$$\begin{array}{cc} & \text{Tina} \\ & \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \text{Sol} \begin{array}{c} \text{Heads} \\ \text{Tails} \end{array} & \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \end{array}.$$

5. a. For the game defined by the following matrix, determine the best strategies for both players.

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

b. Add 5 to each element in the matrix given in part a. How does this effect the best strategies?
c. Multiply each element in the matrix in part a by 3. How does this effect the best strategies?
d. Make a conjecture based on the results of parts b and c.
e. Challenge: Use algebra to prove your conjecture.

6. In a game known as Two-Finger Morra, two players simultaneously hold up either one or two fingers. If they hold up the same number of fingers, player 1 will win the sum (in pennies) of the digits from player 2. If they hold up different numbers, then player 2 will win the sum from player 1. Write the payoff matrix for this game. Find the best strategy for each player and the expectation for the row player. Is this a fair game? Explain your answer.

7. In another version of the game in Exercise 6, if the sum of the fingers held out by each player is even, player 1 will win 5 cents. If the sum is odd, player 2 will win 5 cents. Write the payoff matrix for this version. Find the best strategy for each player and the payoff expectation for the row player. Is this a fair game? Explain your answer.

8. A group of parents in a small town in the Midwest are in an uproar about a new social studies program that the school district has adopted. They are seeking to have the program removed from the curriculum. A second group of parents believe the new program is a solid choice and are organizing in favor of keeping it. In order to bring the issue before the voters in the town, the opposing group must collect 400 supporting signatures from registered voters. Both sides are campaigning vigorously by making telephone calls, sending out mailings, and going door to door to contact voters. The local newspaper has estimated the number of signatures that the opposing group is expected to collect with each combination of strategies. What are the best strategies for both groups of parents? If both follow their best strategies, can the opposing group expect to gather enough signatures to get the issue on the ballot? (Hint: Use the concept of dominance to eliminate a row and column.)

		Group in favor		
		Phone	Mail	Door
Group against	Phone	150	75	100
	Mail	350	300	200
	Door	500	100	400

9. Two rival TV networks compete for prime time audiences by showing comedy, drama, and sports. The following matrix gives the payoffs for network A in terms of percentages of regular viewers who watch its channel for various combinations of programs. Find the best strategy for each network and the expectation for network A.

		Network B		
		Comedy	Drama	Sports
Network A	Comedy	10	50	20
	Drama	40	30	50
	Sports	30	20	60

10. In a campaign for student council president at Northeast High the top two candidates, Betty and Bob, are each making two promises about what they will do if they are elected. The payoff matrix in terms of the number of votes Betty will gain follows. What is the best strategy for each candidate and what is Betty's expectation?

$$\begin{array}{cc} & \text{Bob} \\ & \begin{array}{cc} A & B \end{array} \\ \text{Betty} \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 200 & 100 \\ 50 & 180 \end{bmatrix} \end{array}$$

Project

11. The games you studied in this and the previous lesson are known as *zero-sum games*, because one person's loss is the other's gain. If, for example, Sol wins $2, then Tina loses the same amount. In some games, a particular outcome may be worth 2 to one player, but not −2 to the other. Examples of such games include the prisoner's dilemma, chicken, and arms races between countries. Research and report on games that are not zero sum.

Computer/Calculator Exploration

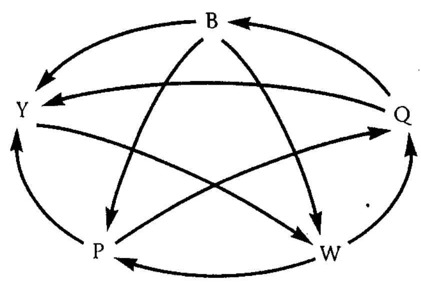
12. Write a graphing calculator (or computer) program using matrices to find the expected payoff for the row player in a two-person game that is not strictly determined.

A Look at a Dominance Matrix

In this example a dominance matrix is used to examine "pecking order" behavior among five neighborhood cats (Bruiser, Wraque, Wruin, Pebbles, and Boy) who enjoy hunting for mice in an open field near their homes. The purpose of using this simple struggle for top cat in the field is to give you an understanding about how this application can be used to measure and compare the dominance of one person over another in political or business situations.

Close observation of the behavior of the cats reveals that there is a definite sense of who is allowed to hunt near the choicest mouse holes when more than one cat is in the field. Bruiser lives up to his name and chases away Wruin, Boy, and Pebbles. Feisty little Wraque stands up to Bruiser and let's Boy know, in no uncertain terms, that she is the boss cat when their paths cross. Wruin dominates her little sister, Wraque, and will not tolerate Pebbles. Pebbles always picks on Wraque and Boy. Finally, Boy, even though he is the smallest, somehow manages to intimidate fat, fuzzy Wruin.

The directed graph on the next page illustrates this furry dominance. The direction indicates who is the dominant cat for each possible pair. Letters *B*, *Q*, *W*, *P*, and *Y* represent Bruiser, Wraque, Wruin, Pebbles, and Boy, respectively.



A 5 by 5 **dominance matrix** D can be used to represent this situation where a 1 represents the dominance of one cat over another and 0s otherwise. The relationship expressed in the dominance matrix is the same as that represented in the digraph.

$$D = \begin{array}{c} \\ B \\ Q \\ W \\ P \\ Y \end{array} \begin{array}{c} \begin{array}{ccccc} B & Q & W & P & Y \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Notice that the entries along the main diagonal in the dominance matrix are all zero. This represents the fact that a cat cannot dominate itself. Notice, also, that if entry D_{ij} is a 1, then entry D_{ji} is a 0. This is true since only one cat can dominate the other. They cannot be mutually dominant. It can also be seen by looking at the matrix that the relationship among the cats is not transitive. For example, Bruiser dominates Wruin, and Wruin dominates Wraque, but Bruiser does not dominate Wraque.

Since the relationship among the cats is not transitive, it is not immediately apparent who is the most powerful. One way to decide this issue would be to look at the dominance matrix and sum the numbers in each row. If we do this, the result is called an **authority vector**:

$$A = \begin{array}{l} \text{Bruiser} \\ \text{Wraque} \\ \text{Wruin} \\ \text{Pebbles} \\ \text{Boy} \end{array} \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

This places Bruiser at the top of the pecking order in dominating three of the other cats. Wraque, Wruin, and Pebbles tie with dominance over two each, and poor Boy lands at the bottom with the least amount of power. This *direct* measure may be all right for a determining the top cat in the neighborhood. However, if we were using this system to examine the relative power or influence over others among a group of people, we would probably be interested in indirect measures of influence as well. For example, if B has direct influence over P and P has direct influence over Q, then there is a sense that B may have some *indirect* influence over Q. This indirect, or second-order, influence should be taken into consideration in determining the degree of B's power.

Return now to the dominance matrix D for this group of cats. Suppose you square this matrix:

$$D^2 = \begin{array}{c} \\ B \\ Q \\ W \\ P \\ Y \end{array} \begin{array}{c} \begin{array}{ccccc} B & Q & W & P & Y \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

What do the entries in this matrix represent? Look at the entry in row 1 column 2. This value is found by multiplying the components of row 1 of matrix D by the corresponding components in the second column and summing the five products:

$$(0 \times 0) + (0 \times 0) + (1 \times 1) + (1 \times 1) + (1 \times 0) = 1 + 1 = 2.$$

Notice that the entries in D^2 can only be nonzero if both factors in at least one product are equal to 1. In the first nonzero product, the first 1 comes from the fact that Bruiser dominates Wruin, and the second 1 comes from the fact that Wruin dominates Wraque. And since this product is nonzero, we can deduce that Bruiser has an indirect second-order influence over Wraque through Wruin. Similarly, the second nonzero product indicates that Bruiser dominates Pebbles, and Pebbles dominates Wraque. This shows that Bruiser also has an indirect second-order influence over Wraque through Pebbles. Thus the 2 in row 1, column 2 of D^2 indicates that Bruiser has two second-order influences over Wraque. We can verify this conclusion by looking at the digraph. One edge points from B to W and another points from W to Q. A second indirect path points from B to P and from P to Q.

You can use a similar argument to show that D^3 will represent the number of third-order influences that exist for each cat.

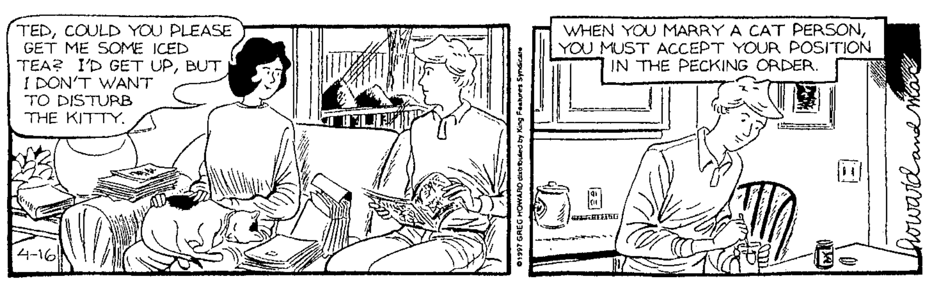
$$D^3 = \begin{array}{c} \\ \text{B} \\ \text{Q} \\ \text{W} \\ \text{P} \\ \text{Y} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{B} & \text{Q} & \text{W} & \text{P} & \text{Y} \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 1 & 3 \\ 0 & 3 & 1 & 2 & 1 \\ 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 0 & 2 \end{bmatrix} \end{array}$$

Now if you add all the entries in corresponding rows from the three matrices D, D^2, and D^3 (omitting the diagonal of matrix D^3), the result will be an influence vector showing the total number of direct first-order and indirect second- and third-order influences exercised by each cat over the others:

$$I = \begin{array}{l} \text{Bruiser} \\ \text{Wraque} \\ \text{Wruin} \\ \text{Pebbles} \\ \text{Boy} \end{array} \begin{bmatrix} 15 \\ 10 \\ 10 \\ 9 \\ 5 \end{bmatrix}$$

The resulting vector indicates that Bruiser is still top cat and Boy remains at the bottom, but at least one of the ties has been eliminated. The question now is where do you stop when you are adding up dominance matrices in this way? Also, is it perhaps more reasonable to give less weight to the second-, third-, and higher-order influence matrices than we give to the direct-order matrix? One way to accomplish this is by dividing D^2 by 2 and D^3 by 3, and so on, if higher-order influence matrices are used in determining the power of each individual member of the group.

The case of the dominant cat may not be of great interest to persons other than the cat owners. However, the method developed in the cat problem can be applied to groups of people such as legislators or corporate leaders to determine who is the most influential or who has the most power. In these situations, one person may show support for another by voting for a bill or merger that he or she has sponsored. A 1 in the dominance matrix is assigned to the person who receives the most support from the other.



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Chapter 7 Review

1. Write a summary of what you think are the important points of this chapter.

2. Suppose that a three-sector economy has the consumption matrix

$$C = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.1 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \end{bmatrix} \end{matrix}.$$

   a. Draw a weighted digraph for this economy.
   b. A production matrix, P, follows. Find the internal consumption matrix product CP. Find the external demand matrix D, where $D = P - CP$.

$$P = \begin{bmatrix} 8 \\ 12 \\ 15 \end{bmatrix}.$$

   c. An external demand matrix, D, follows. Find the production matrix P for this economy. Recall that $P = (I - C)^{-1}D$.

$$D = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}.$$

3. Mike and Nancy are playing poker for pennies to kill time during lunch. Mike is holding a very poor hand and is considering bluffing or not bluffing. Nancy can either call or not call the bluff. The payoff matrix for this situation is shown on the next page. Over the course of

several games in which Mike comes up with a poor hand, what should his strategy be?

		Nancy	
		Call	Not call
Mike	Bluff	−10	10
	Not bluff	−2	0

4. Mike and Nancy soon get bored playing the game described in Exercise 3. They each draw two cards from the deck. Mike draws a 4 of spades and an ace of hearts. Nancy draws a 3 of clubs and a 2 of diamonds. They make up a new game to play with the following rules. Each player will show one card. If both cards shown are the same color, Nancy will pay Mike the sum of the face value of the cards in pennies. If the cards shown are of different colors, Mike will pay Nancy the sum of the face values shown.

a. Find the best strategies for both Mike and Nancy.
b. Use a probability tree to calculate the probability of each of the four possible outcomes when Mike and Nancy play their best strategies.
c. Set up a probability distribution for this game.
d. Find the expected value of the game for Mike.
e. Explain what is meant by the expected value of the game for Mike in this situation.

5. The discrete mathematics teacher has three class starter activities, one of which she uses to begin class every day: a pop quiz, a quickie review, and a small-group problem-solving activity. She never uses the same activity two days in a row. If she gave a pop quiz yesterday, she will toss a coin, and do a quickie review if it comes up heads. If she used a review, she will toss two coins and switch to problem solving if two heads come up. If she did a problem-solving activity, then she will toss three coins, and if three heads come up, she gives a pop quiz again. The transition matrix for this scheme is

$$\begin{array}{c} \\ Q \\ R \\ P \end{array}\begin{array}{c} \begin{array}{ccc} Q & R & P \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{7}{8} & 0 \end{bmatrix} \end{array}$$

a. If the teacher gives a quiz on Monday, what is the probability that she will give another quiz on Friday?
b. In the long run, how often should the students expect that the teacher will start class with a quiz?

c. What activity will the teacher use most often to begin class, and how frequently will she use it?

6. The Super X sells three kinds of sandwiches that many of the students at Southeast High especially like for lunch—Super X Original, Italian Special, and Barbecue Beef. The Super X clerk observed that the same students were coming in for sandwiches for lunch every school day and that the kind of sandwich that each student purchased depended on what he or she had ordered on the previous visit. He conducted a survey and found that of the students who ordered the Original on their last visit, 20% ordered it again the next time, whereas 25% switched to Italian and 55% switched to Barbecue Beef. Of the students who ordered the Italian sandwich the last time, 35% did so again the next time, but 45% switched to the Original and 20% switched to the Barbecue Beef. Of the students who got the Barbecue Beef the last time, 55% ordered it the next time, 20% switched to the Original, and 25% switched to Italian.

a. Set up the transition matrix for this Markov chain.
b. If the same students tend to buy Super X sandwiches for lunch every day, what is the probability that a student who buys the Italian sandwich on Monday will have it again on Wednesday.
c. In the long run, what percentage of the orders will be for the Original? For the Italian? For the Barbecue Beef?
d. How will access to this information help the Super X clerk?

7. A certain economy consists of three industries: transportation, petroleum, and agriculture. The production of \$1 million worth of transportation requires an internal consumption of \$0.2 million worth of transportation, \$0.4 million worth of petroleum, and no agriculture. The production of \$1 million worth of petroleum requires an internal consumption of \$0.3 million worth of transportation, \$0.2 million worth of petroleum, and \$0.3 million worth of agriculture. The production of \$1 million worth of agriculture requires an internal consumption of \$0.3 million worth of transportation, \$0.2 million worth of petroleum, and \$0.25 million worth of agriculture.

a. Draw a weighted digraph for this economy.
b. Write a consumption matrix, *C*, representing this information.
c. On what sector of the economy is transportation the most dependent? The least dependent?
d. If the agriculture sector has an output of \$5.4 million dollars, what is the input in dollars from petroleum? From agriculture?

e. A production matrix, P, in millions of dollars follows. Find the internal consumption matrix product CP and the external demand matrix D.

$$P = \begin{array}{l} \text{Transportation} \\ \text{Petroleum} \\ \text{Agriculture} \end{array} \begin{bmatrix} 20 \\ 25 \\ 15 \end{bmatrix}$$

f. An external demand matrix, D, in millions of dollars, follows. How much must each sector produce to meet this demand?

$$D = \begin{array}{l} \text{Transportation} \\ \text{Petroleum} \\ \text{Agriculture} \end{array} \begin{bmatrix} 4.6 \\ 5.2 \\ 3.0 \end{bmatrix}$$

8. Two computer companies (1 and 2) are competing for sales in two large school districts (A and B). The following payoff matrix shows the differences in sales for companies 1 and 2 in hundreds of thousands of dollars if they focus their full sales force on either school district. Find the best strategy for each company.

Computer company 2

$$\text{Computer company 1} \quad \begin{array}{c} \\ \text{A} \\ \text{B} \end{array} \begin{array}{c} \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \begin{bmatrix} 3 & 7 \\ -7 & -3 \end{bmatrix} \end{array}$$

9. Suppose that in the final days of a political campaign for mayor in a small midwestern city, the Democrats and Republicans are planning their strategies for winning undecided voters to their political camps. The Democrats have decided on two strategies, plan A and plan B. The Republicans plan to counter with plans C and D. The following matrix gives the payoff for the Democrats of the various combinations of strategies. The numbers represent the percentage of the undecided voters joining the Democrats in each case. Find the best strategies for both parties and the expectation for the Democrats.

Republicans

$$\text{Democrats} \quad \begin{array}{c} \\ \text{Plan A} \\ \text{Plan B} \end{array} \begin{array}{c} \begin{array}{cc} \text{Plan C} & \text{Plan D} \end{array} \\ \begin{bmatrix} 30 & 60 \\ 50 & 40 \end{bmatrix} \end{array}$$

10. A manufacturing company has divisions in Massachusetts, Nebraska, and California. The company divisions use goods and services from each other as shown in the following consumption matrix C.

$$C = \begin{array}{r} \text{Mass.} \\ \text{Neb.} \\ \text{Calif.} \end{array} \overset{\begin{array}{ccc} \text{Mass.} & \text{Neb.} & \text{Calif.} \end{array}}{\begin{bmatrix} 0.04 & 0.02 & 0.03 \\ 0.03 & 0.01 & 0.05 \\ 0.01 & 0.02 & 0.04 \end{bmatrix}}$$

a. Draw a weighted digraph for this situation.
b. Find the total production needed to meet a final consumer demand of \$50,000 from Massachusetts, \$30,000 from Nebraska, and \$40,000 from California.
c. What will the internal consumption be for each division to meet the demands in part b?
d. Suppose there is a increase in consumer demand of \$10,000 from Massachusetts, \$8,000 from Nebraska, and \$12,000 from California. What will be the change in internal consumption and in the total production of goods and services for each division?

11. Two competing dairy stores choose daily strategies of raising, not changing, or lowering their milk prices. The following payoff matrix shows the percentage of customers who go from store A to store B for each combination of strategies. What should each store do?

		Store B		
		Raise	No change	Lower
Store A	Raise	4	−1	−4
	No change	2	1	−2
	Lower	5	2	3

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