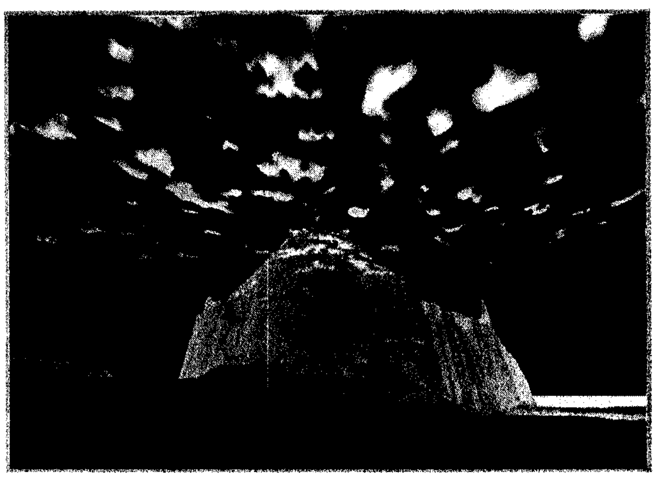
CHAPTER

Recursion

8

Recursion is a process that creates new objects from existing ones that were created by the same process. The recurrence relations you wrote in previous chapters are an example: they enable you to calculate new numbers from existing ones that were calculated with the same formula.

Recursive processes can be geometric, although the computer programs used to implement geometric recursion do so by performing numerical calculations. Fractal images are among the best-known examples of geometric recursion. For example, fractal techniques have been used to create artificial landscapes for science fiction movies.



How can recursion be used to create appealing images? How can recursion help people plan their financial futures? The mathematics of recursion can answer these and other important questions.

Lesson 8.1

Introduction to Recursive Thinking

The simplest recursive processes are numerical: one number in a list is determined by applying simple mathematical calculations to one or more of the preceding numbers. This is the type of recursion you have considered in previous chapters and is also the type with which this chapter begins.

Reconsider a problem that you first saw in Lesson 2.6. Luis and Britt were examining the number of handshakes that occur when every person in a group shakes hands with every other person. The following is a table similar to the one you made in Lesson 2.6.

Number of People in the Group	Number of Handshakes
1	0
2	1
3	3
4	6
5	10

When a new person entered a group in which everyone had shaken hands, the new person had to shake hands with each of the people who were already in the group. Thus, the number of handshakes in a group of n people is $n - 1$ more than the number of handshakes in a group of

$n - 1$ people. If H_n represents the number of handshakes in a group of n people, this recurrence relation can be expressed symbolically as $H_n = H_{n-1} + (n - 1)$.

Your work with this recurrence relation included writing a formula called a **solution to the recurrence relation** and using mathematical induction to prove the formula correct. In this case, the solution, also called a **closed-form solution**, is $H_n = \frac{n(n-1)}{2}$.

Closed-form solutions are useful because, unlike recurrence relations, they calculate a value directly. You can, for example, find the number of handshakes in a group of ten people without knowing the number of handshakes in a group of nine people. However, closed-form solutions can be difficult to find. If such solutions are found by trial and error, mathematical induction can be used to prove their validity.

There are techniques other than trial and error that can be used to find closed-form solutions. For example, the counting techniques discussed in Chapter 6 are useful for certain kinds of recurrence relations. The handshake problem requires that every pair of people shake hands. In a group of n people, there are $C(n, 2)$ ways of selecting a pair, and so there are $C(n, 2) = \frac{n!}{(n-2)!2!}$ handshakes. But $n! = n(n - 1)(n - 2)!$, so the counting solution is equivalent to the solution you hypothesized and proved in Chapter 2.

The closed-form calculation of the number of handshakes in a group of, say, 100 people is a simple one: $100 \times \frac{99}{2} = 4{,}950$. Obtaining this solution with a recurrence relation requires extending the table on page 408 to 100 rows.

Extending a table to 100 or more rows is a tedious task when done by hand. Fortunately, there are several ways to apply technology to the problem. Indeed, the speed of computer and calculator technology has made recursive methods much more useful today than they were only a few decades ago.

One type of technology that is very useful when working with recurrence relations is a computer *spreadsheet*. A spreadsheet is a matrix consisting of columns labeled with the letters A, B, C, . . . and rows labeled with the numerals 1, 2, 3, ... A particular location in the spreadsheet is called a *cell* and is denoted by its column letter and row number, such as A1 or C5. Cells may contain verbal information, numeric values, or formulas based on references to other cells. Spreadsheets have copy features that allow formulas to be copied into other cells so that tables can be generated rapidly.

Another way technology can be applied to recurrence relations is by developing a calculator or computer program that generates values from the relation. Programming requires that an appropriate algorithm be adapted to the language used by the calculator or computer. The following is an algorithm for the handshake problem that can be adapted to a calculator or computer. The variable N represents the number of people in the group, and H represents the number of handshakes.

1. Store the number 1 for variable N and the number 0 for variable H.
2. Display N and H.
3. Add 1 to N and store the result as the new value of N.
4. Add $N - 1$ to H and store the result as the new value of H.
5. Repeat steps 2 through 4.

Step 4 of this algorithm used the recurrence relation to calculate the number of handshakes. The closed form could also be used in this step. To do so, replace step 4 with "store $\frac{N(N-1)}{2}$ as the new value of H."

Finally, some calculators have special functions designed to generate values from recurrence relations.

Using Computer Spreadsheets with Recurrence Relations

To create a spreadsheet for the handshake problem, type suitable labels in the first row. Type initial values of 1 for the number of people in cell A2 and 0 for the number of handshakes in cell B2. In cell A3, type the formula A2 + 1. In cell B3, type the formula B2 + A2, which is equivalent to the recurrence relation $H_n = H_{n-1} + (n - 1)$. The remaining rows are filled by copying row 3. Note that most spreadsheets require the initial character of a formula to be either + or =. The completed spreadsheet is shown here in two ways: with formulas and with numeric results.

	A	B	C
1	Number of people	Number of handshakes	Closed form
2	1	0	=A2*(A2−1)/2
3	=A2+1	=A2+B2	=A3*(A3−1)/2
4	=A3+1	=A3+B3	=A4*(A4−1)/2
5	=A4+1	=A4+B4	=A5*(A5−1)/2
6	=A5+1	=A5+B5	=A6*(A6−1)/2
7	=A6+1	=A6+B6	=A7*(A7−1)/2
8	=A7+1	=A7+B7	=A8*(A8−1)/2
9	=A8+1	=A8+B8	=A9*(A9−1)/2
10	=A9+1	=A9+B9	=A10*(A10−1)/2

	A	B	C
1	Number of people	Number of handshakes	Closed form
2	1	0	0
3	2	1	1
4	3	3	3
5	4	6	6
6	5	10	10
7	6	15	15
8	7	21	21
9	8	28	28
10	9	36	36

Writing Programs for Recurrence Relations

On the left is a computer algorithm written in BASIC that generates a table for the handshake problem. On the right is a similar calculator algorithm for Texas Instruments graphing calculators.

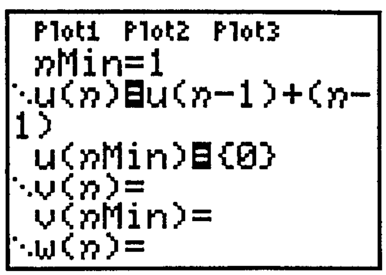
```
10 N = 1:H = 0
20 PRINT N, H
30 N = N + 1
40 H = H + N − 1
50 GO TO 20
```

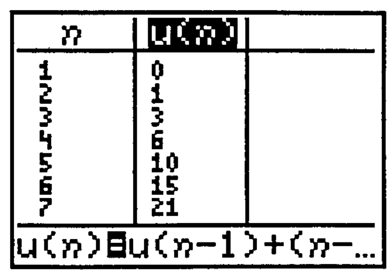
```
1→N:0→H
Lbl A
Disp N, H
N+1→N:H+N−1→H
Goto A
```

Because these algorithms do not end, a statement should be added to terminate the table at some value. For example, to stop the program after calculation of the number of handshakes in a group of ten people, add the line 45 IF N>10 THEN STOP to the BASIC algorithm or the two lines If N>10 and Stop before the last line of the calculator algorithm. Note that the calculator algorithm does not display paired values of N and H on a single line. One way to remedy this inconvenience is store the values in a 1 × 2 matrix and display the matrix.

Calculators with Recursion Features

The following screens demonstrate the recursion features of one type of graphing calculator. The left screen shows the entry of the handshake recurrence relation, which is done after the calculator is placed in its sequence mode. The right screen shows the resulting table, which appears after the table's initial value and increment have been set.





Exercises

1. Consider a variation of this lesson's handshake problem. There are an equal number of men and women in Luis and Britt's group and each person shakes hands with all members of the opposite sex.

a. Draw a graph for a group of four couples in which the vertices represent the men and women in the group and the edges represent the handshakes. Recall your work in graph theory. What kind of a graph is this?

b. If there are only one man and one woman in the group, how many handshakes will be made? With two couples? With three couples?

c. Complete the following table to investigate the number of handshakes that are made.

Number of Couples	Number of Handshakes	Recurrence Relation
1		
2		
3		
4		
5		

d. Assume that there are H_{n-1} handshakes for $n - 1$ couples and that another couple joins the party. How many additional handshakes are now possible?

e. Write a recurrence relation that describes the relationship between the number of handshakes (H_n) for n couples and the number of handshakes (H_{n-1}) for $n - 1$ couples.

2. Consider another variation of the handshake problem in which each man shakes hands with each of the women *except* his date.

a. Make a table showing the number of handshakes that occur when there is one couple. Two couples. Three couples. Four couples.

b. Assuming you know the number of handshakes with $n - 1$ couples, how many additional handshakes are made when the nth couple arrives?

c. Write a recurrence relation for the total number of handshakes (H_n) when there are n couples.

3. a. Write a recurrence relation to describe each of the following patterns. (Note: Do not give closed-form formulas.)

i. 1, 4, 7, 10, 13, ...   ii. 1, 2, 4, 8, 16, 32, ...

iii. 1, 3, 6, 10, 15, 21, 28, ...   iv. 1, 2, 6, 24, 120, 720, ...

b. What is the next term in each of the patterns in part a?

4. The ability to recognize patterns is considered a mark of intelligence. Therefore, most IQ tests include questions about numerical patterns. For example, a question on an IQ test gives the sequence 1, 2, 3, 5, 8, 11 and asks which of the numbers does not belong.

a. Explain why the correct answer is that the last number, 11, does not fit the pattern.

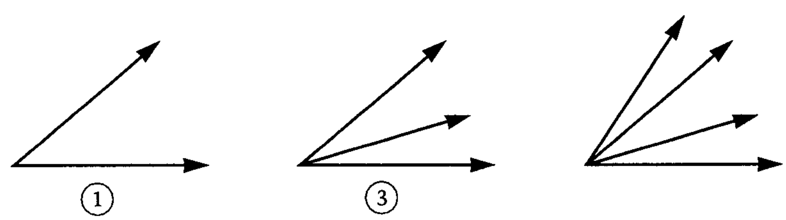
b. Write a recurrence relation that describes the pattern.

5. You cannot use a recurrence relation to generate terms unless you have an initial value. For example, $t_n = 2t_{n-1} - 3$ has terms 5, 7, 11, 19, . . . if the initial value t_1 is 5. But if the initial value is 6, then the terms are 6, 9, 15, 27, ... Notice, however, if the initial value is 3, then all the terms are 3. An initial value for which all the terms of the recurrence relation are the same is called a **fixed point**.

Find the fixed point for each of the following recurrence relations if one exists.

a. $t_n = 2t_{n-1} - 4$.
b. $t_n = 3t_{n-1} + 2$.
c. $t_n = 2t_{n-1}$.
d. $t_n = t_{n-1} + 3$.

6. If two rays have a common endpoint, one angle is formed. If a third ray is added, three angles are formed. See the following figure.



a. How many angles are formed if a fourth ray is added? A fifth ray?
b. Write a recurrence relation for the number of angles formed with n rays.
c. Write a closed-form solution.
d. Use your closed-form solution to find the number of angles formed by ten rays.

7. For the original handshake problem in which everyone shakes hands with everyone else, construct a table for one through eight people in the following manner.

First column: term number

Second column: number of handshakes

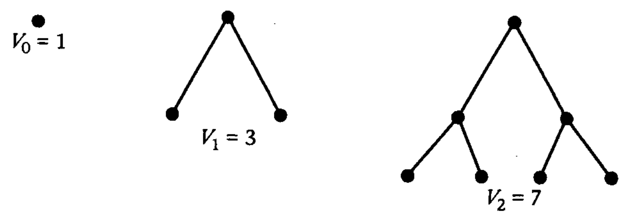
Third column: differences of successive numbers from column 2

Fourth column: differences of successive numbers from column 3

a. What do you notice about the last column?
b. What degree is the polynomial that was obtained for the closed-form solution of the handshake problem? Compare your answer with the number of difference columns in your table.

8. Consider the closed-form polynomial $S_n = 4n^3 - 3n + 2$.
   a. Make a table, as in Exercise 7, for $n = 1, \ldots, 8$. Include difference columns until the numbers in the last difference column are the same.
   b. How many difference columns did you need?
   c. How does the number of difference columns compare with the degree of the closed-form polynomial?

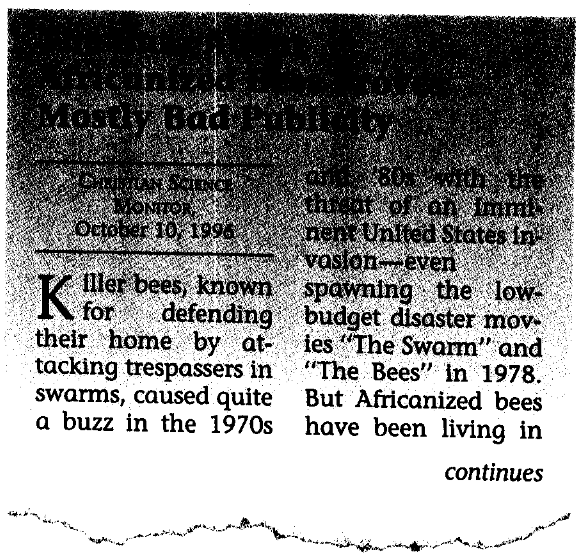
9. Let V_n be the number of vertices in a complete binary tree. (A binary tree is complete if each vertex of the tree has either two or no children.) Level 0 is the root of the tree. The first three trees follow.



   a. Make a table for $V_0, \ldots, V_6$.
   b. Write a recurrence relation to describe V_n.

In Exercise 9, it was convenient to begin the table with V_0 because the root of the tree is considered level 0. There are other cases in which the initial value of the recurrence relation is labeled with the subscript 0. It is, for example, a useful practice when working with time intervals as in Exercises 10 and 11.

10. In the 1980s, residents of the southwestern United States became concerned about an influx of African "killer" bees. In 1987, the number of African bees was estimated at 5,000. It was also estimated that the population would increase at a rate of 12% annually. Let B_n be the number of African bees in



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Mostly Bad Publicity

Christian Science Monitor, October 10, 1996

Killer bees, known for defending their home by attacking trespassers in swarms, caused quite a buzz in the 1970s and '80s with the threat of an imminent United States invasion—even spawning the low-budget disaster movies "The Swarm" and "The Bees" in 1978. But Africanized bees have been living in

continues

The Buzz About Africanized Bees Proves Mostly Bad Publicity (continued)

the Southwest since 1990 without major incident.

Eric Mussen, extension apiculturist at the University of California at Davis, says people are aware to steer clear of the bees and so avoid confrontations. Just recently, a "killer" bee incident was reported in California when tree trimmers were stung by bees that had nested in the tree they were pruning. But despite their bad press, there have been no human or livestock deaths attributed to Africanized bees in the U.S. A 1994 article in *U.S. News and World Report* says people are more likely to be struck by lightning than attacked by Africanized bees.

The Africanized bee arrived in the Western Hemisphere in 1956, when geneticist Warwick Kerr took 75 African queen bees to an apiary near Rio Carlo, Brazil, with the intent to cross breed them with European bees. Africanized bees were known for producing a lot of honey and thriving during rough conditions.

After 26 colonies were accidentally let loose in 1957, the bees spread throughout South and North America, traveling 100 to 200 miles per year.

Nussen says the behavior of pure Africanized bees hasn't changed a lot, but they are inbreeding with the European bee—developing a larger more aggressive hybrid population that is negatively affecting bee colonies in Texas.

Texas each year, where $n = 0$ corresponds to the beginning of the year 1987. Then B_1 would indicate "the end of year 1."

a. Make a table with entries for $B_0, \ldots, B_4$.

b. Write a recurrence relation for B_n.

c. Use a spreadsheet or calculator to determine when the population of African bees is predicted to exceed 100,000. In what year is this predicted to occur?

11. Susie put \$500 in a bank account that pays 5% interest per year, compounded yearly. Let n be the number of years she leaves the money in the bank, let A_0 be the initial amount of money (\$500), and let A_n be the amount of money in the bank at the end of n years.

a. By creating a table, find out how many years it will take for Susie's money to double.

b. Write a recurrence relation for this situation.

12. The term *fractal* was invented in the 1970s by Benoit Mandelbrot to describe a class of geometric objects with unusual properties. One of those properties is that a fractal contains parts that resemble the whole. The following sequence of figures demonstrates the construction of such a "self-similar" object.

A short algorithm describes the process used to create the figures:

1. Draw a line segment.
2. For each line segment in the current figure, erase the middle third and draw a "peak" by constructing an equilateral triangle having the erased portion as a base.
3. Repeat step 2.

a. How does the length of a segment in one figure of the sequence compare with the length of a segment in the preceding figure?

b. How does the number of segments in one figure of the sequence compare with the number of segments in the preceding figure?

c. Write a recurrence relation that describes the relationship between the total length of the segments in figures n and $n - 1$.

Computer/Calculator Explorations

13. Some computer drawing utilities such as Geometer's Sketchpad support recursion. Use a utility with recursion features to construct figures like those in Exercise 12.

14. The Logo computer language supports recursion. Use Logo to construct figures like those in Exercise 12.

15. Perform this spreadsheet experiment: Type the number 1 in cell A1. In cell A2 type the formula A1, and in cell B2 type the formula A1 + B1. Copy the formula in cell A2 into cell A3; copy the formula in cell B2 into cells B3, C3, D3, ... (Stop at a convenient cell in row 3.) Copy row 3 into several of the rows that follow it. Identify the results and explain why they are produced by this procedure.

Projects

16. Research and report on several current applications of recursion not discussed in this chapter. (You may want to begin your research now and continue it as this chapter progresses.)

Lesson 8.2

Finite Differences

Previous lessons discussed two approaches to the problem of finding a closed-form solution to the handshake problem:

1. Trial and error followed by an induction proof of the hypothesized formula
2. Counting techniques

This lesson considers a technique known as the method of finite differences, which can be used to find a closed-form solution to the handshake problem and a variety of related problems.

Recall that the handshake problem is described recursively by $H_1 = 0$, $H_n = H_{n-1} + (n - 1)$. The following is a table generated by this recurrence relation. The third column contains the differences between successive values in the second column; the fourth column contains the differences between successive values in the third column.

Number of People	Number of Handshakes	Differences	
		First	Second
1	0	—	—
2	1	1	—
3	3	2	1
4	6	3	1
5	10	4	1
6	15	5	1
7	21	6	1
8	28	7	1

The constant second differences indicate that the closed-form solution for this recurrence relation is a second-degree polynomial: $an^2 + bn + c$.

Consider what happens when the general second-degree polynomial is evaluated for consecutive integral values of n, and first and second differences are found. The following table shows the results.

Value of n	Value of Polynomial	Differences First	Differences Second
1	$a + b + c$	—	—
2	$4a + 2b + c$	$3a + b$	—
3	$9a + 3b + c$	$5a + b$	$2a$
4	$16a + 4b + c$	$7a + b$	$2a$
5	$25a + 5b + c$	$9a + b$	$2a$

Notice that not only are the second differences constant, the value of the difference is twice the value of the coefficient of n^2. In the case of the handshake problem, this result means that the constant difference of 1 indicates that one term of the closed-form solution is $\frac{1}{2}n^2$.

The remaining terms of the closed-form solution can be found by substituting values from the table into the polynomial $H_n = \frac{1}{2}n^2 + bn + c$.

Although the method just described works well when the closed-form solution is second degree, it is much more tedious for degrees higher than 2. The following alternative method uses technology and is therefore much easier to extend to higher degrees.

Reconsider the handshake problem, a situation in which you know the solution is second degree: $H_n = an^2 + bn + c$. Since there are three values you need to know (a, b, and c), select any three pairs of values from your table. Although any three will do, the first three are the most convenient because of their relatively small values. Form three equations by substituting these three pairs into the general second-degree polynomial $H_n = an^2 + bn + c$.

When $n = 1$, $0 = a + b + c$.

When $n = 2$, $1 = 4a + 2b + c$.

When $n = 3$, $3 = 9a + 3b + c$.

Solve this system using the matrix techniques developed in Chapter 7:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0 \end{bmatrix}$$

The finite differences method can be used whenever the differences in consecutive values of the recurrence relation become constant in a finite number of columns. The degree of the closed-form solution is the same as the number of columns needed to achieve the constant differences. The number of equations in the system needed to find the closed-form solution is 1 more than its degree.

A Finite Differences Example

Consider a stack of cannonballs at Fort Recurrence (see Figure 8.1).

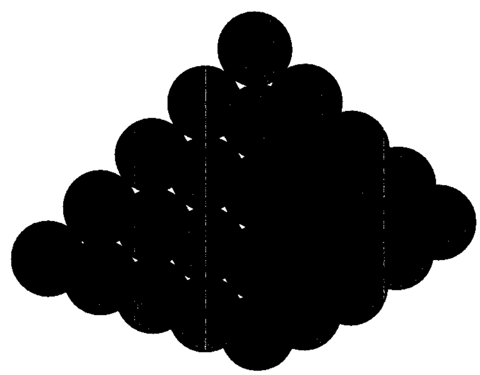


Figure 8.1 Cannonballs at Fort Recurrence.

The following table displays the number of cannonballs in a pyramid of n layers.

Number of Layers	Number of Canonballs	Differences First	Second	Third
1	1	—	—	—
2	5	4	—	—
3	14	9	5	—
4	30	16	7	2
5	55	25	9	2
6	91	36	11	2

The recurrence relation that describes the number of cannonballs in a stack of n layers is $C_n = C_{n-1} + n^2$. The constant differences in the third column indicate that the closed-form solution is third degree: $C_n = an^3 + bn^2 + cn + d$. The system created by this general third-degree polynomial and the first four values in the table is:

When $n = 1$, $1 = a + b + c + d$.

When $n = 2$, $5 = 8a + 4b + 2c + d$.

When $n = 3$, $14 = 27a + 9b + 3c + d$.

When $n = 4$, $30 = 64a + 16b + 4c + d$.

The matrix solution is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 5 \\ 14 \\ 30 \end{bmatrix} = \begin{bmatrix} 0.3333 \\ 0.5 \\ 0.1667 \\ 0 \end{bmatrix}$$

The closed-form solution, therefore, is $C_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. Note that unlike the case in which the solution is second degree, the coefficient of the first term is not one-half the constant difference.

Unfortunately, the finite difference method does not apply to recurrence relations that never achieve constant differences. In such cases, other methods that are described in later lessons of this chapter are often successful. The following exercises investigate several situations that can be described with recurrence relations in which the differences eventually become constant.

Including Differences in a Spreadsheet

If your spreadsheet already contains the number of people and the number of handshakes in columns A and B, then adding columns for differences requires very little effort. A difference column can be added by typing only one new formula and then copying it into as many additional cells as necessary. If, for example, the spreadsheet has the number of handshakes for a group of 1 in cell B2, for a group of 2 in cell B3, and so forth, the first difference is placed in cell C3 by typing the formula B3 − B2. This formula is then copied into other cells of column C. Since the values in column C are not constant, the same formula is copied into the cells of column D starting in cell D4. Because the values in column D are constant, the process stops.

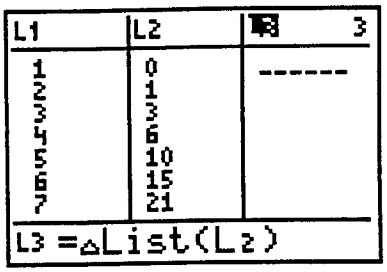
The first spreadsheet following shows the formulas; the second shows the values that result.

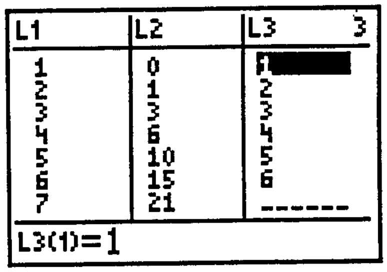
	A	B	C	D
1	Number of people	Number of handshakes	First differences	Second differences
2	1	0		
3	=A2+1	=B2+A2	=B3−B2	
4	=A3+1	=B3+A3	=B4−B3	=C4−C3
5	=A4+1	=B4+A4	=B5−B4	=C5−C4
6	=A5+1	=B5+A5	=B6−B5	=C6−C5
7	=A6+1	=B6+A6	=B7−B6	=C7−C6
8	=A7+1	=B7+A7	=B8−B7	=C8−C7
9	=A8+1	=B8+A8	=B9−B8	=C9−C8
10	=A9+1	=B9+A9	=B10−B9	=C10−C9
11				

	A	B	C	D
1	Number of people	Number of handshakes	First differences	Second differences
2	1	0		
3	2	1	1	
4	3	3	2	1
5	4	6	3	1
6	5	10	4	1
7	6	15	5	1
8	7	21	6	1
9	8	28	7	1
10	9	36	8	1
11				

Difference Columns on a Graphing Calculator

Some graphing calculators have a function that calculates the differences between successive pairs of values in a list. Note that the calculator used to create the following screens places a given difference opposite the first member of the pair rather than the second.





Exercises

1. Use finite differences to determine the degree of the closed-form formula that was used to generate the given sequence.

a. −3, −2, 3, 12, 25, 42, 63, 88, 117, 150, 187, 228, 273, 322, ...

b. 0.29, 0.52, 0.75, 0.98, 1.21, 1.44, 1.67, 1.90, 2.13, 2.36, 2.59, ...

c. 0, −2, −2, 0, 4, 10, 18, 28, 40, 54, 70, 88, 108, 130, 154, ...

d. 1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 111147, ...

2. For each part of Exercise 1, determine the closed-form formula that will generate the sequence.

3. a. Write a recurrence relation for the number of edges T_n in a complete graph with n vertices, K_n.

b. For your recurrence relation in part a, what is the initial condition? (Hint: How many edges are in a graph with one vertex?)

c. Use finite difference techniques to determine a closed-form formula for the number of edges in a K_n graph.

4. $a_1 = 1$ and $a_n = 3a_{n-1} - 5$.

a. Find the first few (six to eight) terms.

b. Find the fixed point for this recurrence relation. (Hint: When a recurrence relation has a fixed point, all the terms are the same. Replace a_n and a_{n-1} with a single variable such as x, then solve. Check your solution by using it as an initial value in the recurrence relation.)

5. A triangle has no diagonal, a quadrilateral has two diagonals, and a pentagon has five diagonals.

a. Write a recurrence relation for the number of diagonals in an n-sided polygon.

b. Use finite difference techniques to find a closed-form formula for the number of diagonals in an n-sided polygon.

6. An auditorium has 24 seats in the front row. Each successive row, moving toward the back of the auditorium, has 2 additional seats. The last row has 96 seats.

a. Create a table with a column for the number of the row and a column for the number of seats in that row. Complete at least the first six entries in the table.

b. Write a recurrence relation for the number of seats in the nth row.

c. Find a closed-form solution for the number of seats in the nth row. (One way to do this is to use finite differences.)

d. How many rows are in the auditorium? Explain.

e. Add a third column, "Total seats," to your table from part a. Complete at least the first six sums in this column.

f. Write a recurrence relation for the number of seats in the first n rows of the auditorium.

g. Write a closed-form formula for the number of seats in the first n rows of the auditorium.

7. A house purchased in 1985 increased in value at the rate of 8% per year.

a. If the original cost of the house was $38,000, calculate the value of the house each year from 1985 to 1998. (A spreadsheet might be nice to use here.)

b. Write a recurrence relation for the value of the house at the end of the nth year since 1985.

c. Calculate the finite differences for your numbers in part a. Do you eventually obtain constant differences?

8. In 1998, a herd of 50 deer is increasing at the rate of approximately 4% per year.

a. Make a table that gives the number of deer at the end of each year ($T_0 = 50$).

b. If the herd's habitat can provide food for a maximum of 325 deer, in what year will there not be enough food?

c. Write a recurrence relation for the number of deer at the end of the nth year.

d. Calculate the finite differences for your table in part a. Do you eventually obtain constant differences?

9. This lesson includes an analysis of second-degree polynomials that discovered a connection between the leading coefficient of a second-degree closed-form solution and the constant difference. Perform a similar analysis for the third-degree polynomial. How is the leading coefficient related to the constant difference?

From Endangered to Dangerous: Once Headed for Extinction, Deer Now are Overpopulated

KANSAS CITY STAR, May 18, 1996

White-tailed deer are the nation's most deadly animal. They cause 120 deaths a year in car crashes—no other animal causes more human deaths.

No animal causes more property damage.

The average damage is more than $600 to each of the 300,000 vehicles a year in deer collisions, says George Harrison in an article he wrote for the current issue of *Sports Afield*. Harrison cites figures from the Insurance Information Institute. The total cost: more than $180 million.

Sixty years ago people were genuinely concerned that deer soon would be extinct. Today, more than 25 million white-tailed deer and 5 million mule deer roam the United States.

And the populations continue to climb. It is estimated the nation has more deer now than at the time of European settlement.

Computer/Calculator Explorations

10. Graphing calculators have statistical functions that fit various kinds of mathematical functions to a set of data. Many of these calculators include several kinds of polynomials in this collection of functions. Prepare a report on the polynomial-fitting capabilities in your calculator and show how they can be used to find closed-form polynomial solutions to recurrence relations for which differences become constant.

Lesson 8.3

Arithmetic and Geometric Recursion

Counting techniques and finite differences are two methods that can be used to find closed-form solutions for recurrence relations. However, there are many kinds of recurrence relations, and no method is capable of finding a closed-form solution for all of them.

Two of the most common types of recurrence relations are those in which a term is generated by either adding a constant to the previous term or multiplying the previous term by a constant. The first type is called **arithmetic**, and the constant is called the **common difference**. The second type is called **geometric**, and the constant is called the **common ratio**. This lesson considers arithmetic and geometric recurrence relations and a few of their many applications.

A surprising fact about these two types of recurrence relations is that a geometric recurrence relation with a common ratio larger than 1 and positive first term will eventually grow to a larger value than an arithmetic recurrence relation, even if the latter's first term and common difference are relatively large. For example, consider a job that employs you for 30 days and in which you have a choice of two methods of payment. Method 1 pays $5,000 the first day and includes a $10,000 raise each day after that. Method 2 pays only $.01 the first day but doubles the amount you are paid each successive day. The questions are, of course, Which salary is better? How much better?

The first method of payment is arithmetic, and the common difference is $10,000. The second method is geometric, and the common ratio is 2. If

P_n is the payment on the nth day, the arithmetic recurrence relation is $P_n = P_{n-1} + 10{,}000$; the geometric recurrence relation is $P_n = 2P_{n-1}$.

To determine the amount by which one salary is better than the other, you must find the total amount each pays over the 30-day period. If T_n represents the total, then a recurrence relation that describes the total is $T_n = T_{n-1} + P_n$.

Of course, all these questions can be answered by using a computer or calculator to generate a comparative table for the entire 30-day period. However, there are formulas that can find all the relevant information directly. Since these formulas have many applications, this lesson next discusses how the formulas are developed.

Formulas for Arithmetic and Geometric Terms

In method 1, the pay on the first day is \$5,000, the pay on the second day is \$5,000 + \$10,000, and the pay on the third day is \$5,000 + \$10,000 + \$10,000 = \$5,000 + 2(\$10,000). Therefore, the pay on the nth day is \$5,000 plus $n - 1$ raises of \$10,000 each. Thus, the formula for the pay on the nth day is \$5,000 + $(n - 1)$\$10,000.

> In general, the nth term of an arithmetic recurrence relation is found by adding $n - 1$ common differences to the first term: $t_n = t_1 + (n - 1)d$.

In method 2, the pay on the first day is \$.01, the pay on the second day is \$.01(2), and the pay on the third day is \$.01(2)(2) = \$.01(2^2). Therefore, the pay on the nth day is \$.01 doubled $n - 1$ times. Thus, the formula for the pay on the nth day is \$.01(2^{n-1}).

> In general, the nth term of a geometric recurrence relation is found by multiplying the first term by the common ratio $n - 1$ times: $t_n = t_1(r^{n-1})$.

With these formulas, the comparison of wages on the 30th day is a simple matter:

Method 1: t_{30} = \$5,000 + (30 − 1)\$10,000 = \$295,000.

Method 2: t_{30} = \$.01(2^{30-1}) = \$5,368,709.12.

The trend lines in Figure 8.2 show the wages for each method over the 30-day period.

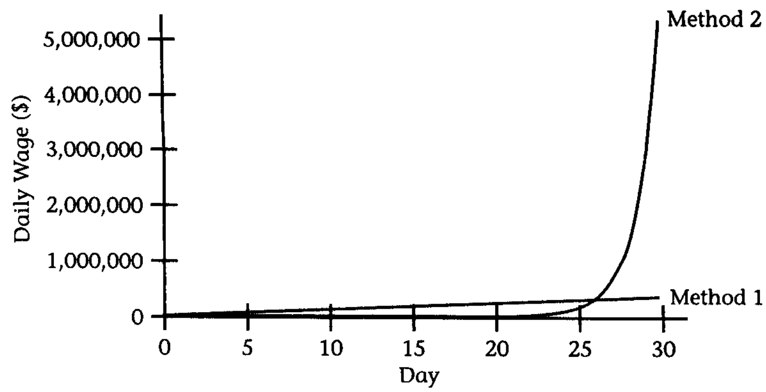


Figure 8.2 Daily wages by two methods.

Sums of Arithmetic and Geometric Terms

A complete comparison of the two methods requires the total of the 30 daily wages for each method. To examine the total pay for method 1, consider the general arithmetic recurrence relation with first term t_1 and common difference d.

Term Number	Term	Sum of First n Terms	Differences: First	Differences: Second
1	t_1	t_1	—	—
2	$t_1 + d$	$t_1 + (t_1 + d) = 2t_1 + d$	$t_1 + d$	—
3	$t_1 + 2d$	$(t_1 + 2d) + (2t_1 + d) = 3t_1 + 3d$	$t_1 + 2d$	d
4	$t_1 + 3d$	$(t_1 + 3d) + (3t_1 + 3d) = (4t_1 + 6d)$	$t_1 + 3d$	d

The constant second differences indicate that the closed-form solution is a second-degree polynomial; $t_n = an^2 + bn + c$. The related system of equations created from the first three pairs in the table is:

When $n = 1$, $t_1 = a + b + c$.

When $n = 2$, $2t_1 + d = 4a + 2b + c$.

When $n = 3$, $3t_1 + 3d = 9a + 3b + c$.

Matrices can be used to solve the system:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} t_1 \\ 2t_1 + d \\ 3t_1 + 3d \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} t_1 \\ d \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0.5 \\ 1 & -0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ d \end{bmatrix} = \begin{bmatrix} 0.5d \\ t_1 - 0.5d \\ 0 \end{bmatrix}.$$

In general, the sum of the first n terms of an arithmetic recurrence relation is $0.5dn^2 + (t_1 - 0.5d)n$.

The sums in method 2 do not generate constant differences in a finite number of steps, and, therefore, a closed-from solution cannot be found by the finite difference method. It can, however, be found by algebraic means.

Consider the general geometric recurrence relation with first term t_1 and common ratio r. The sum of the first n terms, S_n, is

$$S_n = t_1 + t_1r + t_1r^2 + \cdots + t_1r^{n-1}.$$

Multiply this equation by r:

$$rS_n = r(t_1 + t_1r + t_1r^2 + \cdots + t_1r^{n-1}).$$

Distribute r on the right side of the equation, and subtract this equation from the original equation:

$$S_n = t_1 + t_1r + t_1r^2 + \cdots + t_1r^{n-1}$$

$$rS_n = \qquad t_1r + t_1r^2 + t_1r^3 + \cdots + t_1r^{n-1}$$

$$S_n - rS_n = t_1 - t_1r^n.$$

Now factor both sides and divide by $(1 - r)$:

$$S_n(1 - r) = t_1(1 - r^n),$$

or

$$S_n = \frac{t_1(1 - r^n)}{(1 - r).}$$

In general, the sum S_n of the first n terms of a geometric recurrence relation is

$$\frac{t_1(1 - r^n)}{(1 - r)}.$$

Multilingual Software Starts to Open Doors in Cyberspace

London, (CBS MarketWatch), May 13, 1998

The multilingual jungle of the Internet has created demand for translation software that allows users to quickly read web pages in their own tongues, and that has the companies behind the software among the most popular exhibits at Internet World 98 this week in London.

Orange, California-based newcomer LanguageForce launched its new Deluxe Universal Translator at the London conference. LanguageForce uses word-for-word technology developed in Russia and says its software can translate documents, e-mail, and WWW pages in 33 different languages, including everything from Arabic and Chinese to Japanese and Swahili.

In 1997, only its second year, the company had revenues of more than $6 million, said president Ian Simpson. "But we're reaching for $11 million in 1998." And he says that $16 million–$20 million is likely in 1999. The overall translation market, including the old-fashioned kind that uses humans with dictionaries, will be worth about $11.2 million in 1998; and that is projected to grow to around $17.2 million in 2003, Simpson says, meaning that his company should see "geometric growth," as translation software grabs an increasing share of the pie.

With the arithmetic and geometric sum formulas, comparison of the total wages is a simple matter:

Method 1: $S_{30} = 0.5 \times 10{,}000 \times 30^2 + (5{,}000 - 0.5 \times 10{,}000) \times 30 =$ \$4,500,000.

Method 2: $S_{30} = 0.01(1 - 2^{30})/(1 - 2) =$ \$10,737,418.23.

Including Sums in Spreadsheets

If your spreadsheet has the term numbers and the terms of a recurrence relation in columns A and B, respectively, adding a column for the sum requires entering a single formula, then copying it into as many cells as you need. For example, if the first term of the recurrence relation is in cell B1, the second term is in B2, and so forth, type B1 in cell C1 and the formula C1 + B2 in cell C2. Copy the formula into cells C3, C4, and so on.

Including Sums in Programs

Lesson 8.1 gave an algorithm for generating terms of the recurrence relation $H_n = H_{n-1} + (n - 1)$ and implementations for the BASIC language and a calculator language. To include sums in the algorithm, introduce a variable (S) for the sum, give it an initial value equal to the first term, and have it accumulate values of the terms as the terms are generated. Following are Lesson 8.1's implementations with instructions to accumulate sums.

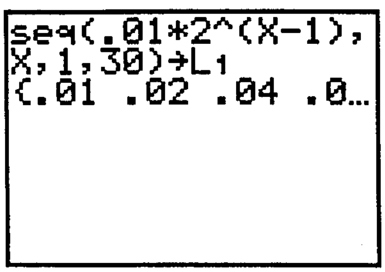
```
10 N = 1:H = 0:S = H
20 PRINT N, H, S
30 N = N + 1
40 H = H + N − 1
50 S = S + H
60 GO TO 20
```

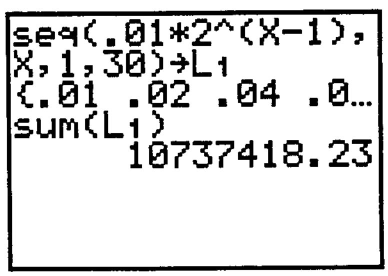
```
1 → N:0 → H:H → S
Lbl A
Disp N, H, S
N + 1 → N:H + N − 1 → H:S + H → S
Goto A
```

Keep in mind that neither algorithm terminates unless additional instructions are added.

Sums on a Graphing Calculator

Some graphing calculators have functions that automatically calculate sums. The screen on the left shows the closed-form solution $t_n = 0.01 \times 2^{n-1}$ as it is being used to calculate the 30 daily wages by method 2. The results are stored in one of the calculator's lists. The second screen shows the sum function as it is being used to find the sum of the 30 daily wages.





In this lesson's exercises you have the opportunity to apply your knowledge of arithmetic and geometric terms and sums in a variety of situations.

Exercises

1. Consider the following sequences:

i. 2, 5, 8, 11, 14, ...

ii. 64, 32, 16, 8, 4, 2, 1, ...

iii. 10, 12, 14.4, 17.28, 20.736, ...

iv. 2, 3, 5, 8, 13, 21, ...

v. $\frac{3}{10}, \frac{3}{100}, \frac{3}{1,000}, \frac{3}{10,000}, \ldots$

vi. 3, 4, 6, 9, 13, 18, ...

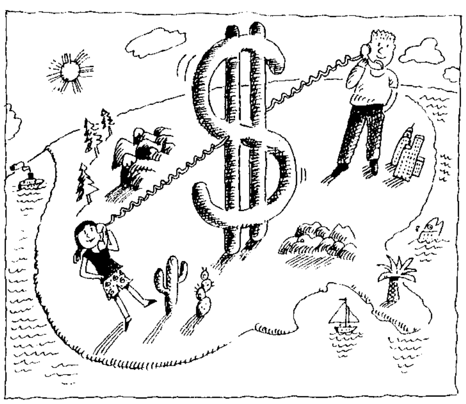
a. Which of the sequences are geometric, which are arithmetic, and which are neither?

b. Write a recurrence relation for each sequence.

c. For those sequences that are arithmetic or geometric, write a closed-form formula for the *n*th term.

2. Consider the general recurrence relation for a geometric sequence, $t_n = rt_{n-1}$. Find the fixed point for this recurrence relation if one exists. (Recall the hint in Exercise 4b, page 424, of Lesson 8.2.)

3. Consider the general recurrence relation for an arithmetic sequence, $t_n = t_{n-1} + d$. Find the fixed point for this recurrence relation if one exists.



4. A teen chat line is a 900 telephone number line that charges \$2 for the first minute and \$.95 for each additional minute.

a. What recurrence relation does the statement of the problem suggest? (Make a table if necessary.)

b. Find a closed-form formula to describe the cost of the call.

c. Assume that 5,000 teens use the line each week and talk for an average of 15 minutes. How much income is produced? Suppose also that the bulk of the cost to the company operating the line is the cost of the long distance line which averages \$3 for a 15-minute call. How much profit does the company make each week?

5. George deposits \$5,000 in the bank at an interest rate of 4.8% compounded yearly.

a. Write a recurrence relation for the amount of money in the account at the end of *n* years.

b. Write a closed-form formula for the amount of money in the account at the end of *n* years.

c. How much money will be in George's account at the end of 3 years?

d. Suppose the bank decides to become competitive with other banking institutions in the city by offering an interest rate of 4.8% that compounds monthly. What monthly interest rate will George receive?

e. Write a recurrence relation for the amount of money in the account at the end of n months.

f. Write a closed-form formula for the amount of money in the account at the end of n months.

g. Find the amount in George's account at the end of 3 years (36 months).

h. Compare the amount of money in George's account at the end of 3 years when the interest is compounded yearly with the amount at the end of 3 years when the interest is compounded monthly.

6. In Exercise 5, suppose George is given the choice of investing his $5,000 at 4.8% compounded monthly or at 5.0% compounded yearly. Compare the two methods of investment at the end of 1 year. At the end of 2 years. At the end of 3 years.

7. Find the fifteenth term and the sum of the first 15 terms for the sequences in parts a to c.

a. 4, 9, 14, 19, ...

b. 45.75, 47, 48.25, 49.5, ...

c. 3650, 3623, 3596, 3569, 3542, ...

d. From your work in other mathematics courses, you may be familiar with the use of the Greek letter Σ (sigma) to indicate sums. For example, if you are summing the first five terms of the sequence whose terms are generated by $2n - 3$, the sum can be indicated in this way: $\sum_{1}^{5}(2n - 3)$. The term numbers of the first and last terms that are included in the sum are written at the bottom and at the top of sigma, respectively, and the formula for the nth term is written in parentheses to the right. Use this type summation notation to represent each of the sums you found in parts a to c.

8. The number of deer on Fawn Island is currently estimated to be 500 and is increasing at a rate of 8% per year. At the present time, the island can support 4,000 deer, but acid rain is destroying the vegetation on the island and the number of deer that can be fed is decreasing by 100 per year. In how many years will there not be

Because of the demise of natural predators, deer populations sometimes threaten to multiply out of control. Discrete mathematical models play an important role in managing deer populations.

enough food for the deer on Fawn Island? Explain how you obtained your answer.

9. The cost of n shirts selling for \$14.95 each is given by $C_n = 14.95n$.

a. What is the equivalent recurrence relation?

b. Is the recurrence relation arithmetic, geometric, or neither? If it is arithmetic or geometric, what are the first term and the common difference or ratio?

10. At 5.5% compounded yearly, what amount must Bill's parents invest for Bill at age 10 if they want Bill to be a millionaire when he reaches age 50?

11. To double your investment at the end of 11 years, what annual interest rate do you need to receive?

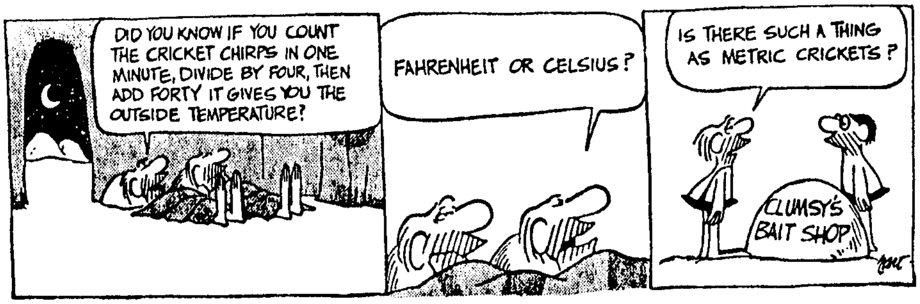
12. The following table gives data relating temperature in degrees Fahrenheit to the number of times a cricket chirps in one minute. Although the data are not quite a perfect arithmetic sequence, they are close to one.

Temperature (°F)	50	52	55	58	60	64	68
Chirps per minute	40	48	60	73	80	98	114

a. Assuming that the data form an arithmetic sequence, with the temperature as the term number and the chirps per minute as the term value, what is the common difference per degree rise in temperature?

b. What is the temperature when a cricket is chirping 110 times per minute?

c. At what temperature does a cricket stop chirping?



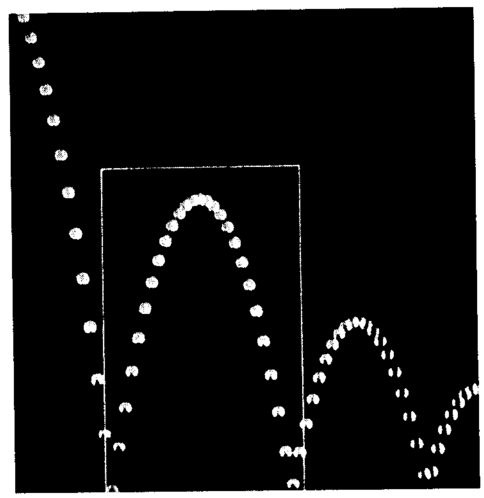
B.C. by permission of Johnny Hart and Field Enterprises, Inc.

13. A ball is dropped from a height of 8 feet and rebounds on each bounce to 75% of its height on the previous bounce.

a. How high does the ball reach after the sixth bounce?

b. What is the total distance that the ball has traveled just before the seventh bounce?

14. The tenth term of an arithmetic sequence is 4, and the twenty-fifth term is 20. Find the first term and the common difference for this arithmetic sequence.



A rubber ball rebounds to a fraction of the height of the previous bounce, as shown in this time-lapse photograph.

Parents, Legislators Ask Why College Tuition Keeps Rising

CHRISTIAN SCIENCE MONITOR, September 29, 1997

Question: What do these items have in common? Hair dryers, restaurant-quality dorm food, faculty salaries, designer student services, and penny-pinching legislatures.

Answer: They're all reasons colleges and universities cite for again hiking tuition and fees above the inflation rate.

American undergraduates will pay 5 percent more this year than last for tuition and fees at a four-year college, according to the College Board's Annual Survey of Colleges, released last week. Inflation is sputtering along at 2.2 percent.

Adjusting for inflation, college tuition has jumped a whopping 90 percent over the past 15 years—higher than health care. Meanwhile, family income increased by only 9 percent over the same period.

The average four-year private college costs $19,213; the average four-year public university, $7,472.

15. The average cost of a year of college education in a public university is about $7,500. The cost is increasing an average of 5% per year.

a. If the current rate continues, what will be the cost of a year of college education in 30 years?

b. Ten years from now you start saving for the college education of a child. The best interest rate you can get is 6% compounded annually. What amount would you have to put in the bank in order to pay for a year of college 20 years later?

16. Over time, the number of bacteria in a culture grows geometrically. There are 600 bacteria at $t = 0$ and 950 bacteria at $t = 3$ hours.

a. What is the approximate common ratio for this sequence?

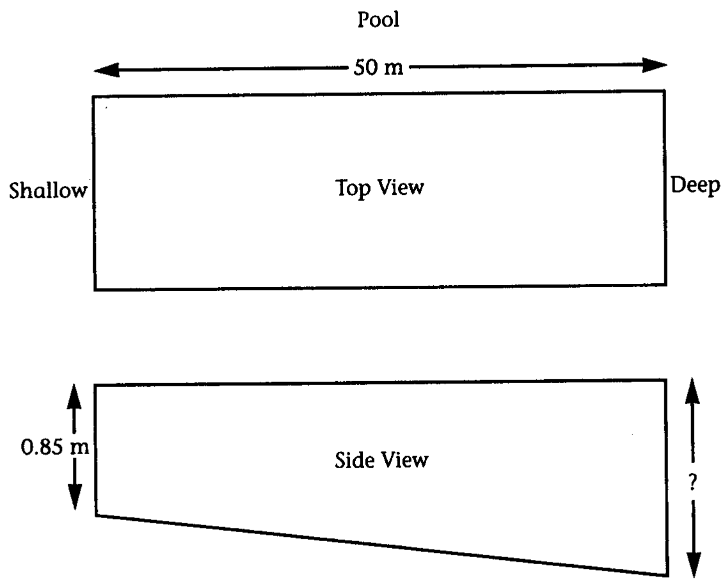
b. If the growth rate continues, approximately how many bacteria will there be in 10 hours?

c. At what time will there first be 50,000 bacteria in the culture?

17. A 50-meter-long pool is constructed with the shallow end 0.85 meter deep. For each meter of length, starting at the shallow end, the pool deepens by 0.06 meter.

a. How deep is the deepest part of the pool? Explain.

b. A rope is to be placed across the pool where the pool depth is 1.6 meters in order to mark the end of the shallow section. How far from the shallow end of the pool should the rope be placed?



18. A company has sales of \$6 million one year and \$8 million the next year. The company expects to experience "geometric growth" at a similar rate in succeeding years. (See the news article on page 430.)

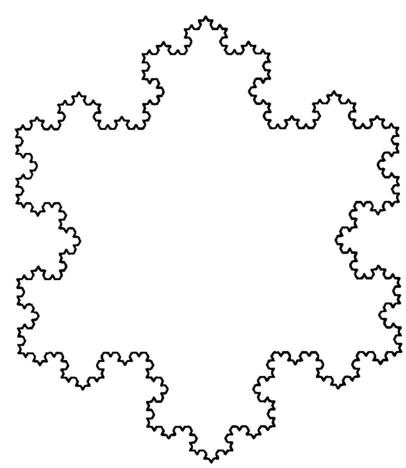
a. Write a recurrence relation that describes the company's growth and find a solution to the recurrence relation.

b. Predict the company's sales 5 years from now and 10 years from now.

c. Do you think the predictions are realistic? Explain.

Computer/Calculator Explorations

19. The fractal curve construction described in Exercise 12 of Lesson 8.1 on pages 416 and 417 can be applied to the sides of an equilateral triangle to produce a fractal that is often called a snowflake curve. At right is one stage of such a construction. Adapt the procedure you developed in either Exercise 13 or Exercise 14 in Lesson 8.1 to produce several snowflake curves.



Projects

20. Some geometric recurrence relations can be used to generate infinite sequences with finite properties. For example, the recurrence relation $t_n = \frac{1}{2} t_{n-1}$ with $t_1 = 1$ generates the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. The terms of this sequence approach 0, and the sequence of sums: $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, \ldots$ approaches 2. Research and report on conditions for which the sequence of terms and the sequence of sums of a geometric recurrence relation approach a finite value. Apply the results of your investigation to the perimeter and area of a snowflake curve based on an equilateral triangle with sides of length 1 (see Exercise 19).

21. In this lesson, the closed-form formulas for terms and sums of arithmetic and geometric recurrence relations are proved by algebraic means. Prepare a report showing how these formulas can be proved by mathematical induction.

Mixed Recursion, Part 1

The previous lesson considered arithmetic and geometric recurrence relations. This lesson examines recurrence relations that have both a geometric and arithmetic component.

To begin, consider a game called Towers of Hanoi. The game involves three pegs and disks of varying sizes stacked from largest to smallest on one of the pegs (see Figure 8.3).

The object of the game is to move the disks from the first peg to the third peg in as few moves as possible. Disks may be placed temporarily on the middle peg or moved back to the first peg, but only one disk may be moved at a time and a disk may never be placed on top of one that is smaller than it.

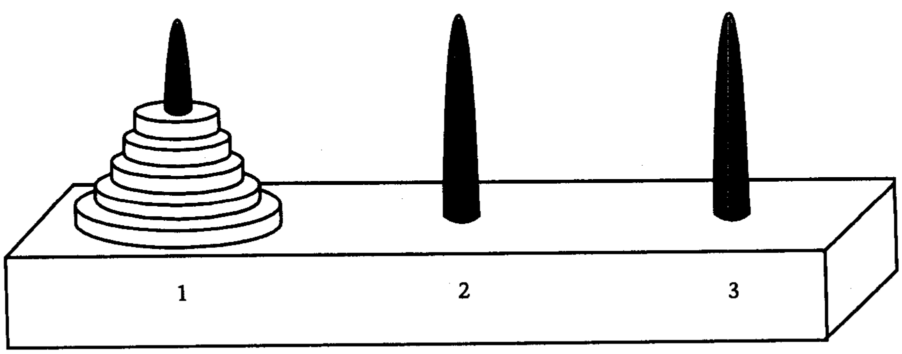


Figure 8.3 Towers of Hanoi.

Artwork appearing on the box for the original Towers of Hanoi puzzle. The puzzle was first marketed in 1883 by the French mathematician Edouard Lucas under the pseudonym "Professor N. Claus (of Siam)." The text translates as "The Tower of Hanoi, Authentic Brain Teaser of the Annamites, A Game Brought Back from Tonkin by Professor N. Claus (of Siam), Mandarin of the College of Li-Sou-Stian!"

Explore This

Cut several round or square pieces of different sizes from paper or poster board. Use the pieces as disks. Number three spots on a large piece of paper and stack several of the pieces you cut from largest to smallest on spot number one. You now have a rudimentary version of the Towers of Hanoi puzzle.

At the direction of your instructor, team up with one or more members of your class and play the game. Keep a record of the fewest moves in which games with different numbers of disks can be completed in a table like the following.

Number of Disks	Fewest Moves to Complete Puzzle
1	1
2	
3	
4	
5	
⋮	

As you progress, look for patterns in the data as well as in the way you are completing the puzzle. If M_n represents the fewest moves in which a game of n disks can be completed, look for a recurrence relation that describes the relationship between M_n and M_{n-1}. If time permits, conjecture a closed-form formula for M_n.

Your investigation of the Towers of Hanoi puzzle should have concluded that neither an arithmetic nor a geometric recurrence relation described the number of moves: you cannot generate a term of the sequence by adding a constant or by multiplying by a constant; you need to do both.

A **mixed recurrence relation** is one in which both multiplication by a constant and addition of a constant are required. The general form of a mixed recurrence relation is $t_n = at_{n-1} + b$. Many common situations can be modeled with mixed recurrence relations. Examples include financial applications such as annuities and loan repayment, the way an object cools or heats, and the way in which diseases spread in a population.

To use mixed recurrence relations, you should be able to recognize that a given set of data can be modeled with a mixed recurrence relation. Consider a table of data generated by the relation $t_n = 2t_{n-1} + 3$ with $t_1 = 4$.

n	t_n	Differences
1	4	—
2	2(4) + 3 = 11	11 − 4 = 7
3	2(11) + 3 = 25	25 − 11 = 14
4	2(25) + 3 = 53	53 − 25 = 28
5	2(53) + 3 = 109	109 − 53 = 56

Note that the values in the difference column grow by a factor of $a = 2$. This characteristic of data generated by a mixed recurrence relation makes it possible to identify situations in which a mixed recurrence relation is an appropriate model and to find the values of a and b.

When data are generated by a mixed recurrence relation, the ratio of any difference to the one preceding it is the same as the value of a in $t_n = at_{n-1} + b$.

For example, consider the following table showing the increase in the value of a house over several years after its purchase.

Year	Value	Differences
0	80,000	—
1	82,000	2,000
2	84,200	2,200
3	86,620	2,420

The ratio of the second difference to the first is $\frac{2{,}200}{2{,}000} = 1.1$. The ratio of the third difference to the second is $\frac{2{,}420}{2{,}200} = 1.1$. A mixed recurrence relation is therefore an appropriate model, and the value of a is 1.1.

The recurrence relation that models the value of the house after n years is $V_n = 1.1V_{n-1} + b$. The value of b can be found by substituting any two successive values from the second column of the table into this equation. For example, if the first two values are used, the result is $82{,}000 = 1.1(80{,}000) + b$. Solving for b gives $b = 82{,}000 - 1.1(80{,}000)$, or $b = -6{,}000$.

The completed model for the value of the house after n years is $V_n = 1.1V_{n-1} - 6{,}000$.

A Mixed Recursion Example

Many people save money by making regular deposits—from each paycheck, for example. Often such accounts are part of a retirement plan that shelters current income from taxes. An account to which you make regular contributions is called an **annuity**. Annuities can be modeled with mixed recurrence relations.

Consider an account that pays 6% annual interest compounded monthly, to which monthly additions of $200 are made. Since the interest is compounded monthly, the model is based on a monthly interest rate of $\frac{0.06}{12} = 0.005$. The following table shows the growth of the account during the first few months.

Month	Balance ($)
0	200
1	1.005(200) + 200 = 401
2	1.005(401) + 200 = 603.01
3	1.005(603.01) + 200 = 806.03

If B_n represents the balance at the end of the nth month, then $B_n = 1.005B_{n-1} + 200$. This mixed recurrence relation can be used to create a table that tracks the growth of the account over many months or years.

Annuities on Spreadsheets

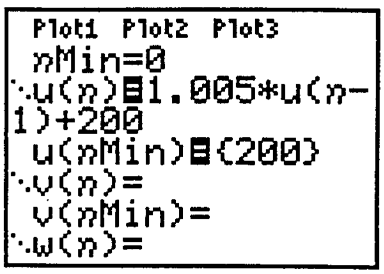
Spreadsheet techniques discussed in previous lessons can be used to create annuity models. However, it is useful to create the spreadsheet in a way that allows monthly payments and interest rates to be changed easily so you can explore various options.

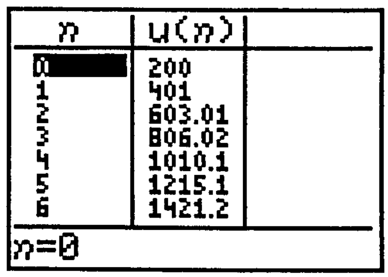
Create a spreadsheet in which a monthly payment of $200 and an annual interest rate of 0.06 are stored in cells A1 and B1. Type column headings of "Month" and "Balance" in the next row. In cell A3 type 0 and in cell B3 type the simple formula A1. In cell A4 type the formula A3 + 1. In cell B4 type the formula (1 + $B$1/12)*B3 + $A$1. This formula is the key to the spreadsheet model. The dollar sign causes the referenced columns and rows to be fixed, which means that the spreadsheet will not change them when the formula is copied into other cells.

	A	B
1	200	0.06
2	Month	Balance
3	0	= A1
4	= A3 + 1	= (1 + $B$1/12)*B3 + $A$1
5		
6		

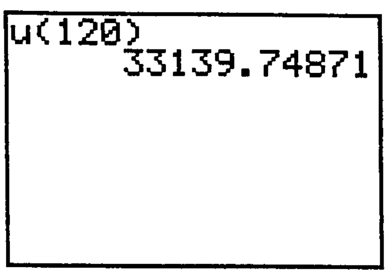
Complete the spreadsheet by copying the formulas in row 4 into as many rows as you like.

Annuities on a Graphing Calculator





The screens above show how the annuity example in this lesson is modeled on one type of graphing calculator and the table that results. The table can be scrolled to any desired value. However, annuities are often active for many years, and scrolling to, say, the tenth year (120th month) can be tedious. This calculator allows calculation of any term on the home screen (below).



Exercises

1. Use the annuity example of this lesson with a monthly addition of $200 and an annual rate of 6% compounded monthly.

a. Extend the table to 20 years (240 months). Record the balance at the end of 20 years.

b. Determine the portion of the balance after 20 years that was paid into the account by its owner.

c. Determine the portion of the balance after 20 years that was paid in interest.

d. Compare the amount in the account after 30 years with the amount after 20 years.

e. Change the interest rate to 8% compounded monthly and determine the amount in the account after 20 years. How does it compare with the balance after 20 years when the interest is 6%?

f. Change the monthly addition to $300 and determine the amount in the account paying 6% after 20 years. How does the balance compare with the balance after 20 years in the 6% account with $200 monthly additions?

2. Following is a table similar to the one you made for the Towers of Hanoi puzzle.

Number of Disks	Number of Moves
1	1
2	3
3	7
4	15
5	31

The recurrence relation that describes these data is $M_n = 2M_{n-1} + 1$. You may have noticed that the number of moves always seems to be 1 smaller than a power of 2 and conjectured the closed-form formula $M_n = 2^n - 1$. In this exercise, assume that this is the correct formula.

a. The original version of the puzzle was accompanied by a legend that at the beginning of time God created a temple in which a group of priests are working tirelessly to complete a 64-disk version of the puzzle. Find the number of moves necessary to complete the 64-disk version.

b. The legend says that the priests move one disk a second and work nonstop in shifts; when the puzzle is completed, God will end the world. Find the number of years from the beginning of creation until the end of the world according to this legend.

3. The closed-formula formula for the Towers of Hanoi problem, $M_n = 2^n - 1$, can be proved by mathematical induction.

a. What initial step is necessary?

b. The proof assumes that the formula works for a puzzle with k disks and then attempts to prove the formula must also work for a puzzle with $k + 1$ disks. Rewrite the assumption and goal of the induction proof in terms of the formula.

c. Show how the proof is completed by applying the recurrence relation.

4. Marty had $2,000 in her savings account when she graduated from college and began work. She converted the savings account to an annuity to which she will deposit $100 each month. She expects the annuity to earn 0.5% monthly.

a. Complete this table.

n (in Months)	t_n
0	2,000
1	1.005(2,000) + 100 = 2,110
2	
3	

b. Find the recurrence relation for the amount of money at the end of n months.

c. Use your recurrence relation to find the amount in Marty's account at the end of the fourth month. At the end of the fifth month.

5. The Hamadas have borrowed $12,000 to buy a boat. The yearly interest on the loan is 8.4% and the monthly payment is $286.

a. Complete the table.

t (in Months)	t_n
0	12,000
1	1.007(12,000) − 286 = 11,798
2	
3	

b. Write a recurrence relation for the loan balance at the end of n months.

c. Use a spreadsheet, calculator, or computer program to extend the table in part a until the loan is paid off. The values that you are calculating form what is known as an *amortization schedule.*

d. Explore what happens to the Hamadas' loan if the interest rate and monthly payment remain the same but they borrow $34,000. What will be the amount in the loan at the end of 1 month? At the end of 2 months? At the end of n months? What is the $34,000 called?

6. Newton's law of cooling says that over a fixed time period the change in temperature of an object is proportional to the difference between the temperature of the object and its surrounding environment. If t is the temperature of the object, s is the temperature of the surroundings, and a is the constant of proportion, then $t_n - t_{n-1} = a(t_{n-1} - s)$.

   A cup of cocoa was brewed to a temperature of 170°F. When set in a room whose temperature is 70°F, the temperature of the cocoa dropped to 162°F in 1 minute.

   a. Write the recurrence relation for the temperature of the cocoa after n minutes.
   b. Simplify the recurrence relation you wrote in part a to the form $t_n = at_{n-1} + b$.
   c. Use the recurrence relation to find the temperature of the hot chocolate after 2 minutes.
   d. Find the fixed point for this recurrence relation. What is the significance of the fixed point in this situation?

7. To help him finish his final year of college, Sam took out a loan of $5,000. At the end of the first year after he graduated, there was a $4,500 balance, and at the end of the second year, $3,950 remained. The amount of money left at the end of n years can be modeled by the mixed recurrence relation $t_n = at_{n-1} + b$.

   a. The information stated above is summarized in the following table:

n	t_n
0	5,000
1	4,500
2	3,900

   Use the general form of a mixed recurrence relation and the data in the table to write a system of equations. Solve for a and b. What is the recurrence relation for the amount of money in Sam's account after n years?

   b. What will be the balance owed on the loan at the end of the third year? At the end of the fourth year?
   c. What is the rate of interest on this loan?
   d. Find the fixed point for this recurrence relation. What is the significance of this amount of money?

8. Suppose a college's tuition over the past 3 years has risen from $8,000 to $8,700 to the present cost of $9,435. Use a mixed recurrence relation of the form $t_n = at_{n-1} + b$ to predict next year's tuition.

9. A virus is spreading through Central High. In the following table, n represents a given period of time, and t_n represents the number of people exposed to the virus at the end of the time period.

n	t_n
5	500
6	750
7	900
8	990
9	1,044

a. Write a recurrence relation that models these data.

b. Use the recurrence relation to find the number of people exposed to the virus at the end of time period 10. At the end of time period 4.

10. Models for exposure to disease often assume that the number of people exposed during a given time interval is directly proportional to the number of people not yet exposed at the beginning of the time interval. In other words, if t_n represents the number of people exposed during time period n, P represents the total population, and k is the constant of proportion, then $t_n - t_{n-1} = k(P - t_{n-1})$.

a. Find values of k and P that show this recurrence relation is equivalent to the one you found in Exercise 9.

b. What is the fixed point for your recurrence relation in Exercise 9? What is the significance of the fixed point in this situation?

Flu Epidemic Is Latest Plague to Hit Olympics

MILWAUKEE JOURNAL SENTINEL, February 18, 1998

In an Olympic Games plagued for a week by bad weather, the latest enemy is a vicious flu that has swept through Japan and landed squarely in the lungs of dozens of athletes, officials and journalists in Nagano.

Nearly 900,000 people have taken ill and at least four children have died in Japan this winter in one of the worst influenza outbreaks in years. At least 20 people, including 17 schoolchildren and three elderly people, have died from complications caused by the flu.

Among the athletes affected by the flu are: German figure skater Tanja Szewczenko, who was forced to withdraw from the women's competition; Canadian pairs figure skaters Marie-Claude Savard-Gagnon and Luc Bradet, who finished 16th; Norwegian speedskater Aadne Sondral, who won the gold medal in the 1,500-meter race but had to withdraw from the 1,000 meters; and Canadian silver medalist Elvis Stojko, who blamed a "brutal flu" for his failure to win the gold in figure skating.

Some of the worst-hit people in Nagano have been the 8,000 journalists covering the Games. Most of the journalists work together in the Main Press Center, a huge convention center, and live in apartment complexes constructed for the Games. They travel back and forth on crowded shuttle buses, which seem to be perfect incubators for the flu.

11. The terms of a recurrence relation can be graphed on a rectangular coordinate system in which term numbers are placed on the x-axis and the values of the terms on the y-axis.
   a. Prepare such a graph for the mixed recurrence relation $t_n = 0.5t_{n-1} + 1$ with $t_1 = 4$. Show at least the first ten terms.
   b. What does the graph tell you about the long-term behavior of the recurrence relation?

Computer/Calculator Explorations

12. People often use annuities to save for a goal. Consider someone who wants to have $10,000 at the end of 5 years and expects to earn 6% annual interest compounded monthly. What monthly deposit should this person make? Use a spreadsheet to experiment with different monthly deposits until you find one the does the job and report to your class on the results.

13. Automobile manufacturers occasionally have promotions in which the customer is offered a choice between low-interest financing and a cash rebate. Select a promotion of that type that has recently been offered in your area (you may have to call a few dealerships for the information) and a car to which the promotion applies.
   a. Construct a spreadsheet that compares the two options. In other words, track both the low-interest loan on the car's value after the required down payment and a regular loan on the car's value after the rebate and the required down payment. Which is the better choice?
   b. Construct a spreadsheet that compares taking the low-interest loan with paying cash less the rebate for the car. Would it have been better to take the loan and put the cash in an account, such as a certificate of deposit, that pays a relatively high interest rate or to pay cash for the car to get the rebate? (You will need to check with a bank to get current interest rates.)

14. Conduct an investigation into the advantages and disadvantages of short-term versus long-term loans for a car or a house. For example, compare the payments and total cost of a 30-year home loan versus a 15-year home loan.

15. Mortgage companies sometimes offer a home-loan repayment plan with payments every four weeks instead of monthly, resulting in a shorter loan term. Contact mortgage companies in your area for details of such a plan. Compare the total cost of both plans for a typical home loan.

Mixed Recursion, Part 2

The previous lesson considered several real-world situations that can be modeled with a mixed recurrence relation. The long duration of some applications such as annuities often means that tables of several hundred rows must be created. Although calculators and computers make construction of large tables feasible, a general closed-form solution for mixed recurrence relations would further decrease the time required to answer many questions. This lesson develops the closed-form solution for mixed recurrence relations.

As an example, consider the mixed recurrence relation $t_n = 2t_{n-1} - 3$ with $t_1 = 4$. Recall that the fixed point of a recurrence relation can be found by equating t_n and t_{n-1}. In this case, solving the equation $x = 2x - 3$ gives a fixed point of 3. Now consider a table showing the first four terms of this recurrence relation. Notice, as shown in the third column, that subtracting the fixed point from each term produces a sequence of powers of 2.

n	t_n	$t_n - 3$
1	4	1
2	5	2
3	7	4
4	11	8

Therefore, the closed-form solution is $t_n = 2^{n-1} + 3$.

[illegible] Retirement $$ Early

[illegible], May 18, 1998

Mary Wilson is 27 years old and hasn't saved anything for retirement. For that reason alone, she speaks for a large part of her generation.

Wilson, a production assistant at a New York book publisher, said she's still early in her career and doesn't have the luxury of looking that far ahead.

She's not alone in her reluctance to begin planning for her golden years. According to a 1997 study done by the Employee Benefit Research Institute, 25 percent of all people between the ages of 25 and 33 have no retirement savings.

The incentives to start saving in your 20s are nearly overwhelming. Time is certainly on your side, since the power of compounding can make a large sum out of a small investment.

If at age 21 you invest $2,000 in an individual retirement account each year for five years, assuming 9 percent annual return, you'd have $400,000 by age 65 though your investment was only $10,000.

If you waited until age 40, you could invest $2,000 for 20 years (a total of $40,000) and still only end up with $130,000.

One of the best places to start investing is with a 401(k) plan. Usually you'll have to be at your job for one year before you're eligible, but once you're eligible, your employer will often match a portion of your own contributions.

Drawing a conclusion from a single example is unwise, so consider the same recurrence relation, but this time with $t_1 = 8$. The following table shows the first four terms.

n	t_n	$t_n - 3$
1	8	5
2	13	10
3	23	20
4	43	40

Compare the third column of this table with the third column of the previous table. The powers of 2 are still there, but each has been multiplied by 5, which is the difference between the first term and the fixed point. Therefore, the closed-form solution is now $t_n = 5(2^{n-1}) + 3$. Keep in mind that 5 is the difference between the first term and the fixed point, 2 is the multiplier in the original recurrence relation, and 3 is the fixed point.

If there is uncertainty about the validity of a closed-form formula, mathematical induction can be used to prove that the formula is correct. Consider the closed-form formula $t_n = 5(2^{n-1}) + 3$.

First, be sure the formula produces the correct first term: $5(2^{1-1}) + 3 = 5(1) + 3 = 8$.

Now you must show that if $t_n = 5(2^{n-1}) + 3$ generates the nth term, then $t_{n+1} = 5(2^{(n+1)-1}) + 3 = 5(2^n) + 3$ generates the $n + 1$th term.

The term t_{n+1} is generated by multiplying the previous term by 2 and subtracting 3: $t_{n+1} = 2t_n - 3$, but $t_n = 5(2^{n-1}) + 3$, so by substitution $t_{n+1} = 2[5(2^{n-1}) + 3] - 3$. Simplify this expression:

$$\begin{aligned}&2[5(2^{n-1}) + 3] - 3\\ &= 5(2)(2^{n-1}) + (3)2 - 3\\ &= 5(2^n) + 6 - 3\\ &= 5(2^n) + 3.\end{aligned}$$

Thus, mathematical induction guarantees that $t_n = 5(2^{n-1}) + 3$ will always generate the terms of $t_n = 2t_{n-1} - 3$ with $t_1 = 8$.

It is tempting to base a general conclusion on the previous example: a closed-form solution for a recurrence relation of the type $t_n = at_{n-1} + b$ was $t_n = (t_1 - p)(a^{n-1}) + p$, where p, the fixed point, is $\frac{b}{1-a}$, which is obtained by solving the equation $x = ax + b$. However, the previous induction proof applies only to one specific recurrence relation and, therefore, does not establish a general result. You will do that in this lesson's exercises.

An Annuity Example

Consider the annuity example of Lesson 8.4 (see "A Mixed Recursion Example" on page 442). The account paid 6% annual interest compounded monthly and included monthly deposits of \$200. The recurrence relation for the balance at the end of the nth month is

$$B_n = \left(1 + \frac{0.06}{12}\right) B_{n-1} + 200 \qquad \text{or} \qquad B_n = 1.005\, B_{n-1} + 200.$$

The following table tracks the account for the first few months.

Month	Balance
0	200
1	1.005(200) + 200 = 401
2	1.005(401) + 200 = 603.01
3	1.005(603.01) + 200 = 806.03

Note that $t_0 = 200$, $t_1 = 401$, $a = 1.005$, and $b = 200$. Therefore, the fixed point is $\frac{200}{1 - 1.005} = -40{,}000$

Substituting for t_1, a, and b in the general closed-form solution for mixed recurrence relations, $t_n = (t_1 - p)(a^{n-1}) + p$, gives

$$t_n = (401 + 40{,}000)(1.005^{n-1}) - 40{,}000, \text{ or } t_n = 40{,}401(1.005^{n-1}) - 40{,}000.$$

Determining the amount in the account after, say, 20 years requires evaluating the closed-form solution for $n = 20$:

$$40{,}401(1.005^{20-1}) - 40{,}000 = \$93{,}070.22.$$

A commonly asked question about an annuity involves the time required for it to reach a certain amount. For example, to determine when this annuity will reach \$200,000, solve the equation $200{,}000 = 40{,}401(1.005^{n-1}) - 40{,}000$. Doing so requires a little algebra:

$$40{,}401(1.005^{n-1}) - 40{,}000 = 200{,}000$$

$$40{,}401(1.005^{n-1}) = 200{,}000 + 40{,}000$$

$$(1.005^{n-1}) = \frac{240{,}000}{40{,}401}$$

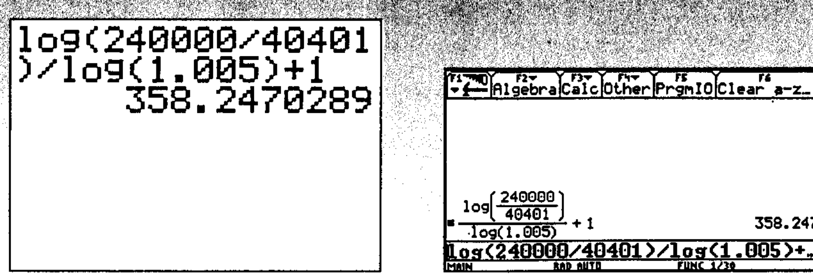
$$(n-1)\log(1.005) = \log\left(\frac{240{,}000}{40{,}401}\right)$$

$$n - 1 = \frac{\log\left(\frac{240{,}000}{40{,}401}\right)}{\log(1.005)}$$

$$n = \frac{\log\left(\frac{240{,}000}{40{,}401}\right)}{\log(1.005)} + 1.$$

The solution can be evaluated on a calculator to obtain about 358 months, or just under 30 years.

Point of Interest



Evaluation of the example solution on two graphing calculators. The expression is typed the same way on both calculators, but the second displays it the way it is usually written.

In the following exercises, the general closed form for mixed recurrence relations is applied to situations similar to those you considered in Lesson 8.4. Use the closed form to solve the problems, and use the recurrence relation together with either a spreadsheet or calculator to check your answers.

Exercises

1. Find the fixed point and the closed form for each of these recurrence relations. Use the closed form to find the 100th term.

a. $t_1 = 1$, $t_n = 2t_{n-1} + 3$. b. $t_1 = 5$, $t_n = 3t_{n-1} - 7$.

c. $t_1 = 2$, $t_n = 4t_{n-1} - 5$.

2. An annuity pays an annual rate of 8% compounded monthly and includes monthly additions of $150.

a. Write the recurrence relation for B_n, the balance at the end of the nth month.

b. Use your recurrence relation to build a table showing B_n for $n = 0$, 1, 2, 3, 4.

c. Find the fixed point for the recurrence relation.

d. Use the fixed point and the value of B_1 to write the closed form.

e. Use the closed form to find the account balance at the end of 30 years.

f. Use the closed form to determine the amount of time that it will take the account to grow to $500,000.

g. Suppose the owner of the account would like to it to reach $500,000 in 30 years. What monthly additions are required? (Hint: You must find the value of b in $B_n = \left(1 + \frac{0.08}{12}\right) B_{n-1} + b$. Write an expression for the fixed point, leaving b as an unknown. Write the appropriate closed form, set it equal to 500,000 when $n = 360$, and solve for b.)

3. Jilian has borrowed $10,000 to buy a car. The annual interest rate is 12% compounded monthly and the monthly payments are $220.

a. Write a recurrence relation for the unpaid balance at the end of the nth month.

b. Use the recurrence relation to tabulate the unpaid balance at the end of months 0, 1, 2, 3, and 4.

c. Find the fixed point for the recurrence relation.

d. Find the closed form.

e. Use the closed form to determine the unpaid balance at the end of 2 years.

f. Use the closed form to determine the amount of time needed to repay the loan. That is, set the closed form equal to 0 and solve for n. Round your answer to the nearest whole number of months.
g. Multiply your previous answer by the monthly payment and determine the amount Jilian really paid for her car. What is the total amount of interest that she paid?
h. Suppose Jilian wanted to pay for the car in 3 years. What would her monthly payment be? (See the hint in part g of Exercise 2.)
i. Interpret the fixed point you found in part c. That is, what is the meaning of the fixed point's dollar value in Jilian's situation?

4. In this lesson the closed-form solution for $t_n = at_{n-1} + b$ was given as $t_n = (t_1 - p)(a^{n-1}) + p$, where p is the fixed point $\frac{b}{1-a}$. Mathematical induction was used to prove this formula, but only for a specific case. This exercise uses mathematical induction to prove that this closed form is correct for all mixed recurrence relations.
a. To begin, verify that the closed form works for t_1: Replace n with 1 in $(t_1 - p)(a^{n-1}) + p$ and show that this really is t_1.
b. The next step is to show that whenever the closed form generates the correct value of t_n, it will also generate the correct value of t_{n+1}. To begin this process, write the closed form for t_n and for t_{n+1}.
c. In a mixed recurrence relation, a term is generated by multiplying the previous term by a and adding b. Generate the $(n + 1)$th term by multiplying the closed form for the nth term by a and adding b.
d. Algebraically simplify the previous expression until it matches the closed form for the $(n + 1)$th term. (Hint: You might find it helpful to replace the second occurrence of the fixed point with $\frac{b}{1-a}$, but not the first.)

5. In Exercise 6 from Lesson 8.4 (page 446) you applied Newton's law of cooling to a cup of cocoa that dropped from 170°F to 162°F in 1 minute in a room whose temperature was 70°F.

a. Rewrite the recurrence relation and recalculate the fixed point for the recurrence relation.

b. Write the closed form for the temperature of the cocoa after n minutes.

c. Use the closed form to find the temperature of the cocoa after 5 minutes.

d. Use the recurrence relation to create a table showing the temperature each minute through the first 5 minutes. Compare the last entry of the table with your answer to part c.

e. Would it make sense to try to use the closed form to determine the time needed for the cocoa to freeze? Try to do so. What happens?

6. The following data represent the number of people at Central High who have heard a rumor.

Number of Hours after Rumor Began	Number of People Who Have Heard It
1	80
2	240
3	320
4	360

a. Write a recurrence relation for the number of people who have heard the rumor after n hours.

b. Find the fixed point for the recurrence relation.

c. Use the fixed point to write the closed form.

d. Use the closed form to determine the number of people who have heard the rumor after 10 hours.

e. The way in which rumors and other information spread through a population is similar to the way in which disease spreads through a population. Compare this exercise with Exercises 9 and 10 of Lesson 8.4 (page 447) and explain the significance of the fixed point you found in part c.

7. People often express surprise at figures like those in the news article on page 450.

a. Check the claim in the article for a 21-year-old person who saves \$2,000 a year for 5 years, then stops. Find the account balance when the person is 65 and explain how you used your knowledge of recurrence relations.

b. Check the claim for the person who starts saving at age 40. Be sure to explain how you used your knowledge of recurrence relations.

c. If a person has saved \$400,000 for retirement, what can the person withdraw each month without decreasing the account balance if the account is earning 8% annual interest compounded monthly when the person retires?

8. The fixed point of a mixed recurrence relation cannot be calculated if $a = 1$ because the denominator of $\frac{b}{1-a}$ is 0. Discuss what to do when this happens.

Projects

9. Design your own savings plan. State the age at which you plan to start saving, the age at which you would like to retire, and the amount of money you would like to have in your account on retirement. Contact investment specialists in your area to determine an approximate rate of return on current annuities. Determine the amount you need to pay into the account on a regular basis to achieve your goal. Discuss the amount you will be able to withdraw from the account to meet regular expenses without depleting the account and if the account is depleted gradually over a reasonable life expectancy. (Keep a copy of the results. One day you will be glad you took this course.)

Cobweb Diagrams

In the last two lessons of this chapter you studied mixed recurrence relations primarily from a numerical viewpoint. Your understanding of these recurrence relations will benefit from a visual technique for representing their behavior.

As an example, consider the recurrence relation $t_n = 2t_{n-1} - 1$ with $t_1 = 3$. The first four terms are shown in the following table.

n	t_n
1	3
2	5
3	9
4	17

Although it may seem obvious, it is important to realize that a given term can be thought of as t_n or t_{n-1}. The first term 3, for example, is t_n if you are thinking of n as 1, but it is t_{n-1} if you are thinking of n as 2. As you build a table of values for a recurrence relation, each term first takes on the role of t_n, then the role of t_{n-1}, and then fades into history, something like an officer of an organization who serves a term as president, then a term as past president, and finally disappears from office. This succession of terms can be visualized by graphing the recurrence relation $t_n = 2t_{n-1}$ as separate functions $y = t_n$ and $y = 2t_{n-1} - 1$, or $y = x$ and $y = 2x - 1$. Associate the x axis with t_n and the y axis with t_{n-1}.

The following algorithm describes the process of graphing the generation of successive terms.

1. Graph the lines $y = x$ and $y = 2x - 1$.
2. Locate the first term of the recurrence relation on the x axis.
3. Draw a vertical line from the first term to the line $y = 2x - 1$. You can think of this step as representing the substitution of the first term into $2t_{n-1}$ in order to generate the second term.
4. Draw a horizontal line from $y = 2x - 1$ to $y = x$. You can think of this line as representing the term's transition from the role of t_n to the role of t_{n-1} in preparation for the generation of the next term.
5. Draw a vertical line upward from $y = x$ to $y = 2x - 1$.
6. Repeat steps 4 and 5 for as many terms as desired.

The graph that results is sometimes called a cobweb diagram (see Figure 8.4).

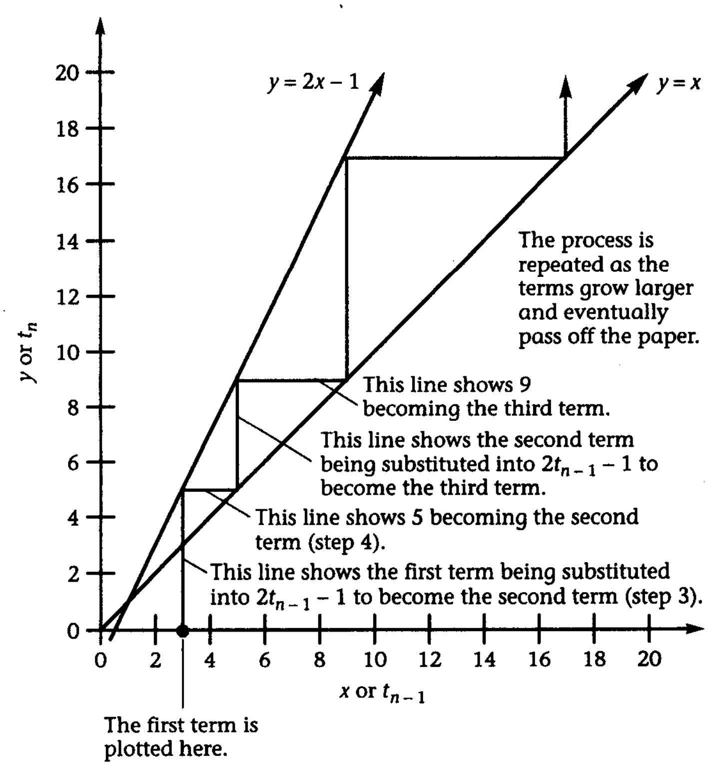


Figure 8.4 Cobweb diagram for the recurrence relation $t_n = 2t_{n-1} - 1$ with $t_1 = 3$.

Note that the intersection of the lines $y = x$ and $y = 2x - 1$ represents the fixed point. If the first term is 1, the first vertical line segment hits the point of intersection and the process can go nowhere from there.

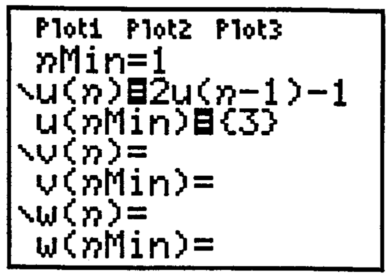
Cobwebs on Graphing Calculators Without Recursion Features

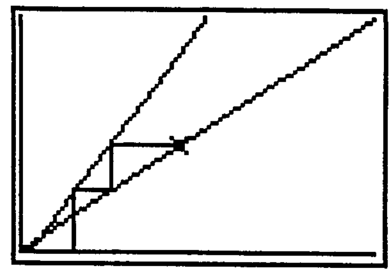
The following is a generic cobweb diagram algorithm that can be adapted to any graphing calculator.

1. Set a suitable graphing range.
2. Graph the line $y = x$ as the calculator's first function ($Y1$) and the line whose equation is found by replacing t_n with y and t_{n-1} with x in the recurrence relation as the calculator's second function ($Y2$). (Be sure all other functions and plots are turned off.)
3. Input the value of the initial point for variable A.
4. Replace X with the value of A.
5. Draw a line from the point $(X, 0)$ to $(X, Y2)$
6. Pause and wait for the user to press ENTER.
7. Draw a line from $(X, Y2)$ to $(Y2, Y2)$.
8. Pause and wait for the user to press ENTER.
9. Replace X with the current value of $Y2$.
10. Draw a line from (X, X) to $(X, Y2)$.
11. Pause and wait for the user to press ENTER.
12. Go back to Step 7.

The first three steps can be included in the program or performed before the program is run. The specific commands to implement the algorithm vary with the calculator model. Consult your calculator manual or talk with someone knowledgeable about your calculator's programming features if you are unsure of what to do.

Cobwebs on Graphing Calculators with Recursion Features





Some graphing calculators have a special graphing mode for cobweb diagrams. With the calculator in the sequence mode, the recurrence relation is entered (left screen). After the calculator is placed in a special web graphing mode, it draws the line $y = x$ and a line representing the recurrence relation (right screen). A segment of the cobweb is drawn each time the user presses a designated key.

The behavior exhibited in this lesson's cobweb diagram example is not the only kind of behavior that can occur. The following exercises explore some other kinds of behavior.

Exercises

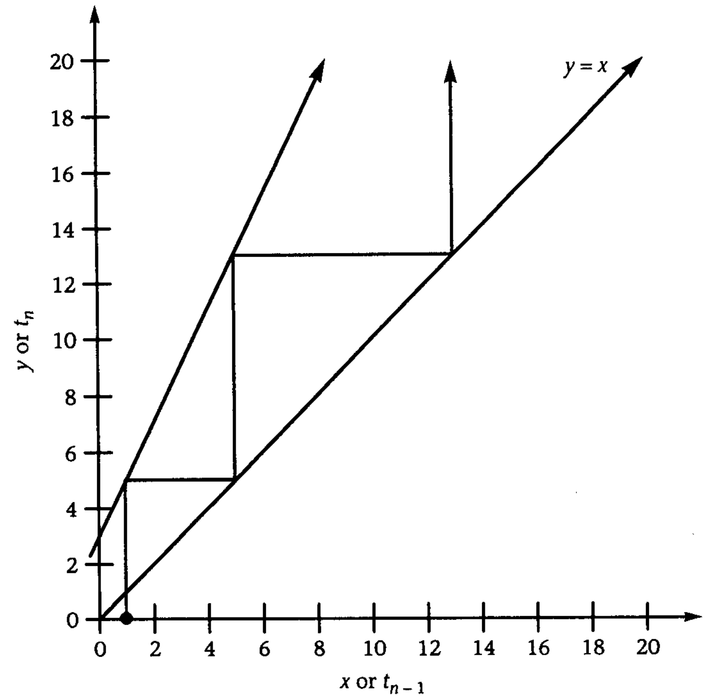
1. Construct a cobweb diagram for the indicated number of terms of each recurrence relation. Find all the terms before beginning the graph so that you can choose a suitable scale for the axes.

a. $t_n = 3t_{n-1} - 8$ with $t_1 = 5$, four terms.
b. $t_n = 5 - t_{n-1}$ with $t_1 = 3$, four terms.
c. $t_n = 9 - 0.5t_{n-1}$ with $t_1 = 2$, four terms.
d. $t_n = 0.5t_{n-1} + 6$ with $t_1 = 2$, four terms.

2. Find the fixed point for each of the recurrence relations in Exercise 1 and mark the fixed point on your cobweb diagram. Mathematicians categorize some fixed points as *repelling*, and others as *attracting*. Which fixed points in Exercise 1 do you think are attracting. Which are repelling? Which appear to be neither?

3. Experiment with mixed recurrence relations $t_n = at_{n-1} + b$. Try various values of a and b. When is a fixed point attracting and when is it repelling? When is it neither?

4. The following figure is a cobweb diagram for a recurrence relation.



a. What is the first term of the recurrence relation?
b. How many terms can be determined from the diagram?
c. Make a table showing all the terms that can be read from the diagram.
d. Write the recurrence relation.
e. Find the fixed point of the recurrence relation from the graph. Show how algebra can be used to find the fixed point.

5. The deer population, estimated at 12,000, in a region has grown so large that the deer are becoming pests. Wildlife biologists estimate that the population is growing at a rate of 4% a year. Officials want to issue a sufficient number of hunting permits so that the population is decreased to 10,000 over the next 10 years.
a. Recommend a number of permits that will accomplish this goal.
b. If officials want to hold the population constant when it reaches 10,000, how many permits should they issue?
c. Do you think this is a reasonable plan? Explain.

6. Consider the recurrence relation $t_n = 4 - 0.5(t_{n-1})^2$ with $t_1 = 0$. This recurrence relation is called *second degree* because a term is generated by squaring the previous term.
   a. Make a table showing the first 6 terms.
   b. Draw a cobweb diagram displaying the first 6 terms.
   c. Change the first term to 1. Use a spreadsheet or calculator to explore the behavior of the first 20 terms. Also explore the related cobweb diagram.
   d. Change the first term to 2. What happens?
   e. Change the first term to 5. What happens?
   f. Use algebra to find the fixed points. (Because this recurrence relation is second degree, it has two.)
   g. You have tried this recurrence relation with first terms of 0, 1, 2, and 5. In which cases were the terms attracted to a fixed point and in which cases were they repelled? Were there any cases in which neither seemed to happen or in which it wasn't possible to tell?

7. A model that is sometimes used to represent the growth of a population in an environment that is capable of supporting only a limited number of the species says that if the uninhibited growth rate of the population is r (in decimal form) and the maximum number the environment can support is m, then the recurrence relation that describes the total number of the species t_n in a given time period is

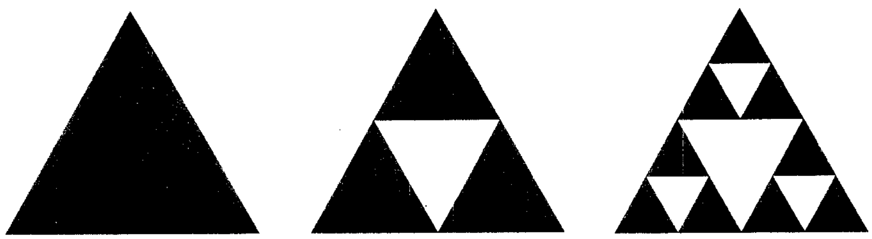
$$t_n = \left[1 + r\left(1 - \frac{t_{n-1}}{m}\right)\right]t_{n-1}.$$

An important idea reflected in this model is that the uninhibited growth rate is reduced as the population approaches the maximum number the environment can support.

The population of a particular species of animal has an annual growth rate of 10%, and the environment is capable of supporting 10 (in thousands) of the animals.
   a. Write the recurrence relation for the number of animals after n years.
   b. Use a table or a cobweb diagram to explore the growth of the population if it currently is 5,000. Describe the results.
   c. Use your table or cobweb diagram to experiment with different initial populations. Be sure to include one that is over 10. Describe the results.
   d. What are the fixed points of the recurrence relation? Explain their significance in this situation.

Computer/Calculator Explorations

8. A fractal related to the snowflake curve (see Exercise 19 in Lesson 8.3 on page 437) is called a Sierpinski gasket. Like the snowflake curve, it is based on an equilateral triangle. It is constructed by connecting the midpoints of the sides of the triangle and removing the triangle that results. The same process is applied to the remaining triangles, and so forth. The first three stages follow.



Use a computer drawing utility or Logo to develop a recursive procedure that draws several of these figures. Investigate their properties. What, for example, can you conclude about the areas and perimeters of these figures?

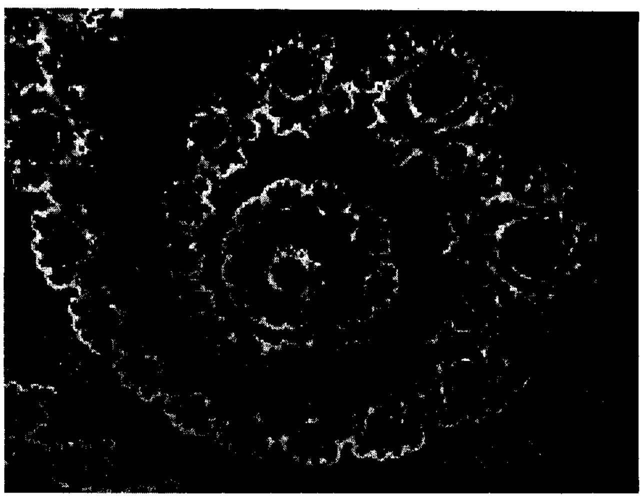
Projects

9. The *order* of a recurrence relation is the difference between the highest and lowest subscripts. Nearly all the recurrence relations in this chapter are first order. The recurrence relation $t_n = t_{n-1} + t_{n-2}$ is second order because the difference between n and $n - 2$ is 2. Investigate some recurrence relations of order higher than 1. Find some applications of these recurrence relations to include in your report.

10. A recent mathematical topic related to mixed recurrence relations is called *chaos*. Prepare a report on this topic. Include the role of mixed recurrence relations and a few applications in your report.

Chapter Extension

Fractal Dimensions



Fractal art is familiar to almost everyone. Its appealing images adorn tee shirts, posters, company logos, and advertisements. But fractals often do their work more discreetly: they are used to create realistic landscapes for movies, to compress data for storage in computers, to estimate the lengths of coastlines, and to create models of biological structures.

A fractal is a figure whose dimension is not a whole number. This fact seems strange to anyone who has never thought about the possibility of a fractional dimension. To understand the concept, it's necessary to think about some familiar objects in a slightly different way.

Consider a line segment, a square, and a cube. Their dimensions are 1, 2, and 3, respectively. Subdivide their sides into, say, 3 equal parts (see Figure 8.5).

Each division forms several objects similar in shape to the original: the segment is divided into 3 segments, the square is divided into 9 squares, and the cube is divided into 27 cubes. The number of similar pieces can be counted by raising 3 to a power: $3^1 = 3$ segments, $3^2 = 9$ squares, and $3^3 = 27$ cubes. The exponent is the same as the figure's dimension.

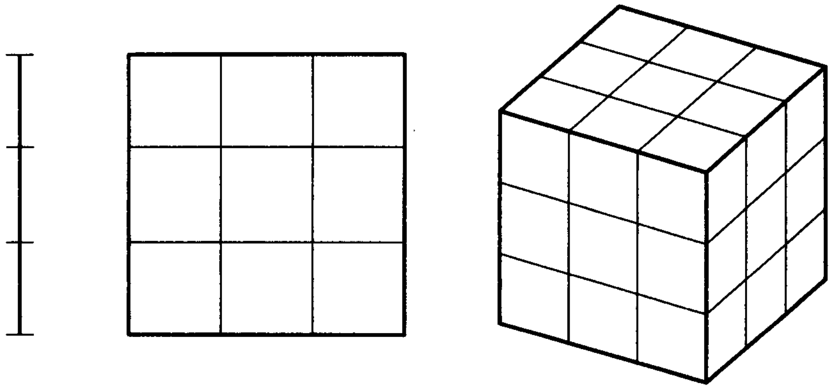


Figure 8.5 Dividing a segment, square, and cube.

Consider a similar analysis of the snowflake curve. The first three steps of its construction are shown in Figure 8.6.

Figure 8.6 The first three steps of the snowflake curve construction.

The second stage is constructed by dividing the first stage into 3 equal pieces, eliminating 1 of them, and adding 2 new ones. The second stage has 4 parts similar to the first stage; the same can be said of the third compared to the second. The fractal dimension is the power d of 3 that equals 4: $3^d = 4$. Solving for d gives the fractal dimension of the snowflake curve:

$$3^d = 4$$

$$d \log 3 = \log 4$$

$$d = \frac{\log 4}{\log 3}.$$

The snowflake curve's dimension is approximately 1.26.

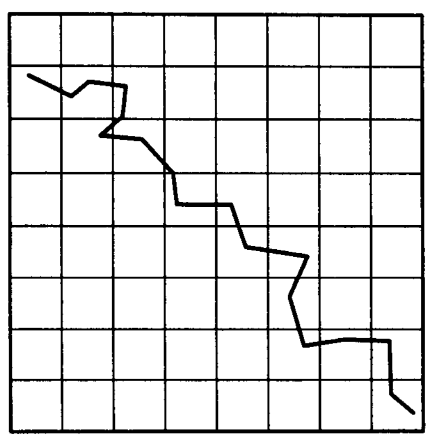
This procedure works for the snowflake curve because it contains copies of itself, in other words, because it is self-similar. To find the fractal dimension of an irregular object such as a coastline, a different procedure is used.

Place a large square over a map of the coastline. Bisect the sides of the square and draw lines dividing the square into 4 smaller squares. Count the number of smaller squares that the coastline intersects. Find the quotient of the log of the number of small squares the coastline intersects and the log of the number of segments into which the sides of the original square are divided (2 in this case).

The process is repeated with a larger number of divisions of the sides of the original square. The ratio

$$\frac{\log(\text{number of squares containing coastline})}{\log(\text{number of segments})}$$

approaches the fractal dimension of the coastline as the number of divisions increases. (Coastline dimensions are usually between 1.15 and 1.25.) Figure 8.7 shows one step in the application of the method.



The large square is divided into $8 \times 8 = 64$ smaller squares. The shape passes through 15 of the smaller squares. Therefore, the quotient is $\frac{\log 15}{\log 8} \approx 1.30$.

Figure 8.7 A step in the calculation of a fractal dimension.

Chapter 8 Review

1. Write a summary of what you think are the important points of this chapter.

2. For each of the following, write a recurrence relation to describe the pattern, find a closed-form solution, and find the 100th term.
   a. 2, 6, 10, 14, . . .   b. 3, 8, 23, 68, . . .   c. 3, 6, 12, 24, . . .

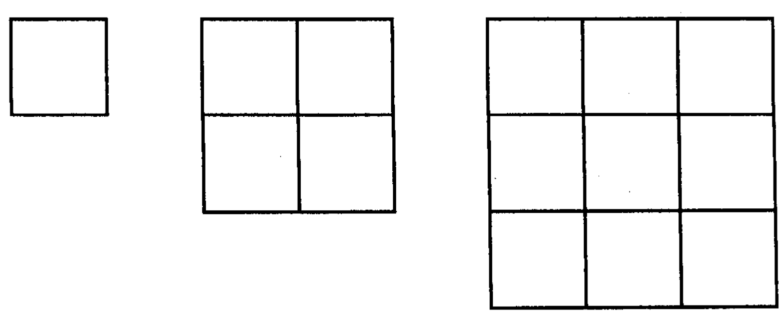
3. Which of the sequences in Exercise 2 are arithmetic? Which are geometric?

4. Find the fixed point for each of the following recurrence relations if one exists. Check your fixed point by using it and the recurrence relation to write the first four terms.
   a. $t_n = 5t_{n-1}$.   b. $t_n = 5t_{n-1} + 3$.   c. $t_n = t_{n-1} - 3$.

5. In Exercise 4, find a closed-form formula for each recurrence relation if the first term is 2. Use the closed form to find the 100th term.

6. Which of the recurrence relations in Exercise 4 are arithmetic? Which are geometric?

7. Consider the following sequence of squares.



a. Let S_n represent the total number of squares of all sizes in the figure whose sides are n units long. For example, $S_2 = 5$ because there are 4 small squares and 1 large one in the second figure. Complete the following table by counting squares and drawing additional figures in the sequence.

n	S_n
1	1
2	5
3	
4	
5	

b. Add difference columns to your table.

c. Find a closed-form formula for S_n. Use your formula to find the total number of squares of all sizes on a checkerboard.

8. Use finite differences to determine the degree of the closed-form polynomial that was used to generate this sequence and show how to find the closed form by solving an appropriate system.

$$-5, -2, 3, 10, 19, 30, 43, \ldots$$

9. The following table shows the number of gifts given on the nth day of Christmas and the total number of gifts given through the first n days as described in the song, "The Twelve Days of Christmas."

Day	Gifts on That Day	Total Number of Gifts
1	1	1
2	$1 + 2 = 3$	$1 + 3 = 4$
3	$1 + 2 + 3 = 6$	$4 + 6 = 10$
4		
5		
6		

a. Complete the table through the sixth day.

b. Write a recurrence relation for the number of gifts given on the nth day, G_n, and a recurrence relation for the total number of gifts given through the nth day, T_n.

c. Find a closed-form formula for G_n and for T_n.

10. In 1999, the cost of a first-class letter was \$.33 for the first ounce and \$.22 for each additional ounce or fraction thereof.

a. Write a recurrence relation to describe the amount of postage P_n on a letter that weighs between $n - 1$ and n ounces.

b. Find a closed-form formula for P_n.

11. Roberto deposited \$1,000 in an account paying 4.8% annual interest compounded monthly.

a. Complete the following table showing the balance in Roberto's account at the end of the first few months.

Month	Balance
0	\$1,000
1	
2	
3	

b. Write a recurrence relation for the balance in Roberto's account at the end of the nth month.

c. Find a closed-form formula for the balance at the end of the nth month.

d. Determine the number of years it will take for the amount in Roberto's account to double.

12. Joan deposited \$5,000 in an annuity account to which she will make \$100 monthly additions. The account pays 6.4% annual interest compounded monthly.

a. Complete the following table showing the balance in Joan's account at the end of the nth month.

Month	Balance
0	\$5,000
1	
2	
3	

b. Write a recurrence relation for the amount in Joan's account at the end of the nth month.

c. Find a closed-form formula for the amount in Joan's account at the end of the nth month.

d. Find the balance in Joan's account at the end of 5 years.
e. How long will it take Joan's account to grow to $50,000?

13. Martha has borrowed $11,000 to buy a car. Her loan carries an annual interest rate of 9.6% compounded monthly and her monthly payments are $230.

a. Write a recurrence relation for the unpaid loan balance at the end of the *n*th month.
b. Write a closed-form formula for the unpaid balance at the end of the *n*th month.
c. How long will it take Martha to pay for her car?
d. What is the total amount of interest that Martha will have paid?
e. If Martha wants to pay off the loan in 3 years, what monthly payments would she make?

14. The following data show the change in value of a painting over a period of years.

Year	Value
1	$8,000
2	$16,000
3	$28,000
4	$46,000

a. Write a recurrence relation to describe the value of the painting after *n* years.
b. Find a closed-form formula for the value of the painting after *n* years.

15. Create a cobweb diagram that displays the first four terms of $t_n = 0.5\,t_{n-1} + 12$ with $t_1 = 12$.

16. The absorption and elimination of medicine by the human body can be modeled with a mixed recurrence relation. Suppose a medication has a recommended dosage of 500 milligrams every 4 hours and that a person's body eliminate's about 60% of the medication in a 4-hour period. (Actual rates vary with the medication and the individual's weight and metabolism.)

a. Write a recurrence relation to model the amount of the medication in the person's body after *n* 4-hour periods.
b. Explore the behavior of the medication over several days. Describe your findings.

c. On the basis of your answer to part b, describe what you would expect to find in a cobweb diagram. Construct one to confirm your answer.
d. Doctors sometimes advise a patient to take a double dose the first time. On the basis of your analysis in parts a to c, why do you think that is?

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